Equivalence between operator spreading and information propagation

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Introduction

Operator spreading and the Lieb-Robinson bound

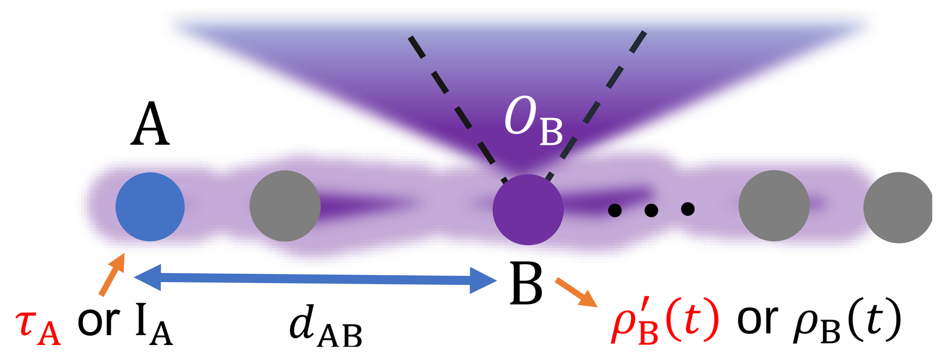

Figure 1: $\varepsilon = \exp\left[-\text{const}\left(d_{AB} - v_{LR}t\right)\right]$, the Lieb-Robinson bound gives an upper bound on operator spreading, where v_{LR} is called Lieb-Robinson velocity[1].

Information propagation and the Holevo capacity

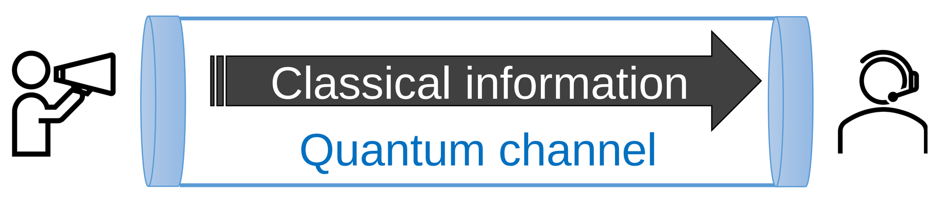

Figure 2: $C_\chi(t)$, the Holevo capacity quantifies the maximum amount of a quantum channel to transmit classical information[2].

Information propagation as operator spreading

$$C_\chi(t) \propto \left|\mathrm{Tr}_B\left\{O_B\left[\rho_B(t) - \rho'_B(t)\right]\right\}\right| \leqslant \sup_{\|O_B\|=1} \left\|O_B(t), \tau_A\right\|_1 \leqslant \varepsilon$$

Simple example:
Encoding: $\{p_0, I_A\}$ and $\{1-p_0, \tau_A\}$, where I_A and τ_A represent identity and unitary operation to A, respectively.
Unitary evolution: $U_{AB}(t) = \mathcal{T}\exp\left[-i\int_0^t H_{AB}(\tau)d\tau\right]$ with any Hamiltonian H_{AB}. We have

$$\rho_B(t) = \mathrm{Tr}_A\left[U_{AB}(t) I_A \rho_{AB}(0) I_A U^\dagger_{AB}(t)\right],$$
$$\rho'_B(t) = \mathrm{Tr}_A\left[U_{AB}(t) \tau_A \rho_{AB}(0) \tau_A^\dagger U^\dagger_{AB}(t)\right],$$

the reduced density matrices $\rho_B(t)$ and $\rho'_B(t)$ correspond to I_A and τ_A.

Finding tight constraints on the classical capacity of quantum channels

1. $C_\chi \propto T$? The trace distance between stationary states ρ_B and ρ'_B can be written as $T(\rho_B, \rho'_B) := 0.5\|\rho_B - \rho'_B\|_1$.
2. $C_\chi(t) \propto \varepsilon$? Case of $\rho_B(t)$ and $\rho'_B(t)$ change over time.
3. How to estimate the classical communication rate, $dC_\chi(t)/dt$?

References: [1] E. H. Lieb and D. W. Robinson, 1972. [2] A. Holevo, 1973.

Setup and main results

Our Setup

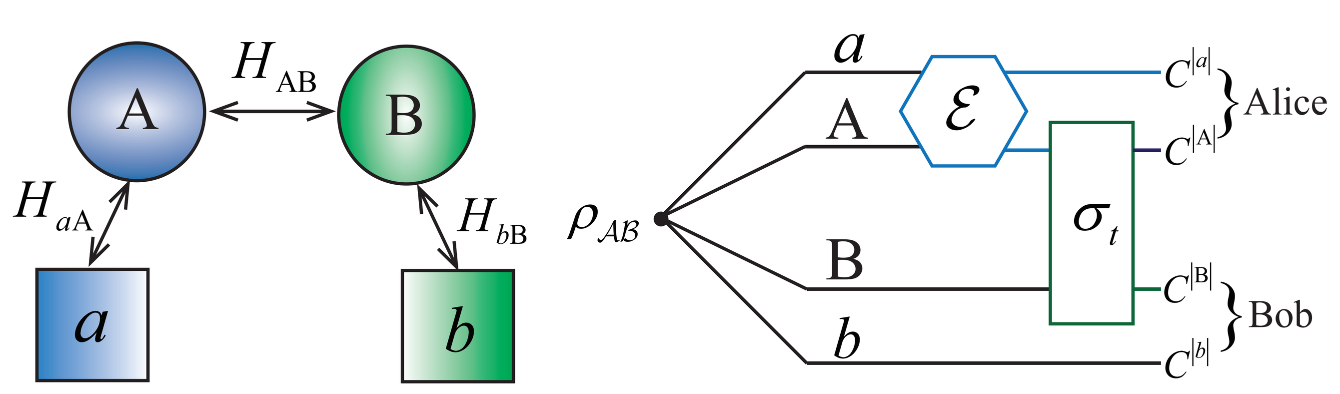

Figure 3: Ancilla-assisted entangling model.

Purification: start from a pure state $\rho_{\mathcal{AB}}$, where $a \cup A = \mathcal{A}$ and $b \cup B = \mathcal{B}$.
Encoding: define $\mathcal{E}$ by a set of CPTP maps acting on $\mathcal{A}$, $\{p_i, \rho_i = \tau_{\mathcal{A},i}(\rho_{\mathcal{AB}})\}$.
A noiseless channel σ_t: $U_{\mathcal{AB}}(t) = I_a \otimes \exp(iH_{AB}t) \otimes I_b$.
we have $\rho^i_{\mathcal{B}} = \mathrm{tr}_{\mathcal{A}}\left[\sigma_t \circ \tau_{\mathcal{A},i}(\rho_{\mathcal{AB}})\right] = \mathrm{tr}_{\mathcal{A}}[U_{\mathcal{AB}}(t)\tau_{\mathcal{A},i}(\rho_{\mathcal{AB}})U^\dagger_{\mathcal{AB}}(t)]$.

Theorem 0. No operator spreading ⇔ zero information propagation

$$C_\chi(t) = \sum_i p_i S\left[\rho^i_{\mathcal{B}}(t) \| \rho_{\mathcal{B}}(t) = p_i\rho^i_{\mathcal{B}}(t)\right],$$

for $\forall i$, $\rho^i_{\mathcal{B}}(t) = \rho_{\mathcal{B}}(t) \Leftrightarrow C_\chi(t) = 0$.

Theorem 1. $C_\chi \propto T\left(\rho^i_{\mathcal{B}}, \overline{\rho^i_{\mathcal{B}}}\right)$?

Using Holevo skew divergence and defining the complementary states $\overline{\rho^i_{\mathcal{B}}}$, the time-independent Holevo capacity is bound by trace distance as

$$\sum_i 2p_i(1-p_i)^2 T\left(\rho^i_{\mathcal{B}}, \overline{\rho^i_{\mathcal{B}}}\right)^2 \leqslant C_\chi \leqslant -\sum_i p_i \log(p_i)\, T\left(\rho^i_{\mathcal{B}}, \overline{\rho^i_{\mathcal{B}}}\right),$$

By applying the improved Pinsker's inequality and defining the measured relative $S_M\left(\rho^i_{\mathcal{B}} \| \overline{\rho^i_{\mathcal{B}}}\right)$, we obtain $T\left(\rho^i_{\mathcal{B}}, \rho^i_{\mathcal{B}}\right) \leqslant \sqrt{1 - \exp\left[-S_M\left(\rho^i_{\mathcal{B}}, \overline{\rho^i_{\mathcal{B}}}\right)\right]}$.

Theorem 2. $C_\chi(t) \propto \varepsilon$?

Information dynamics means operator spreading, which is limited by the Lieb-Robinson bound. The time-dependent Holevo capacity is bound by LR bound

$$\sum_i 2p_i(1-p_i)^2\varepsilon^2 \leqslant C_\chi(t) \leqslant H(p)\varepsilon$$

where the Shannon entropy $H(p)$ is expressed as $H(p) = -\sum_i p_i \log(p_i)$.

Entanglement capacity

Theorem 3. Entanglement capacity, $dC_\chi(t)/dt$?

We prove that it is captured by the small-incremental-entangling theorem as

$$|dC_\chi(t)/dt| \leqslant 2\Gamma_t = 2\mathcal{O}(1)\|H_{AB}\| \log(d_B),\ d_B \leqslant d_A$$

where Γ_t is entangling rate between $\mathcal{A}$ and $\mathcal{B}$. This result holds for all dimensions d_a, d_b and all states $\rho_{\mathcal{AB}}$ of a composite system $\mathcal{AB}$.

Extension 0-2. α Holevo capacity

By introducing a sandwiched Rényi relative entropy, theorem 0 can be extended to

$$C^\alpha_\chi(t) = \sum_i p_i \mathcal{D}_\alpha\left[\rho^i_{\mathcal{B}}(t) \| \rho_{\mathcal{B}}(t)\right],\ \alpha \in (0, +\infty),$$

where

$$\mathcal{D}_\alpha\left[\rho^i_{\mathcal{B}}(t) \| \rho_{\mathcal{B}}(t)\right] = \frac{1}{1-\alpha}\log \mathrm{Tr}_{\mathcal{B}}\left\{\left[\rho^i_{\mathcal{B}}(t)^{\frac{1-\alpha}{2\alpha}} \rho_{\mathcal{B}}(t) \rho^i_{\mathcal{B}}(t)^{\frac{1-\alpha}{2\alpha}}\right]^\alpha\right\}.$$

The limiting value of $\mathcal{D}_\alpha$ as $\alpha \to 1$ back to S. For $C_\chi \propto T\left(\rho^i_{\mathcal{B}}, \overline{\rho^i_{\mathcal{B}}}\right)$ and $C_\chi(t) \propto \varepsilon$, we obtain conclusions similar to Theorems 1 and 2.

Extension 3. α entanglement capacity

The generalized entanglement capacity depends on the Rényi order as

$$|dC^\alpha_\chi(t)/dt| \leqslant \begin{cases} \frac{4\alpha}{|\alpha-1|}\|H_{AB}\| d_B^2, & \alpha < 1, \\ 2\mathcal{O}(1)\|H_{AB}\|\log(d_B), & \alpha = 1 \\ \frac{4\alpha}{|\alpha-1|}\|H_{AB}\| d_B^{\frac{2(\alpha-1)}{\alpha}}, & \alpha > 1, \end{cases}.$$

Future perspective: Extend our results to noisy channels

References: Cheng Shang, Hayato Kinkawa, and Tomotaka Kuwahara, unpublished 2024.

Acknowledgments

Cheng Shang acknowledges the financial support by the CSC and the MEXT Scholarship under Grant No. 211501. Cheng Shang was also supported by the RIKEN Junior Research Associate Program.