Approximate degree lower bounds for oracle identification problems

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Query model

Given: oracle (black box) access to $x \in \{0, 1\}^n$ **Goal**: compute f(x)**Cost:** # of queries

Why consider?

- Strips away implementation details.
- Can prove unconditional lower bounds!



Resources in query model: example

Types of queries:

- Deterministic
 - No additional resources
 - Without error
 - Computing OR requires $\Theta(n)$ queries
- Randomized
 - Access to unbiased random bits
 - Correct with success Pr ⅔
 - Computing OR requires $\Theta(n)$ queries
- Quantum : $\Theta(\sqrt{n})$
 - Query in superposition
 - Correct with success Pr 3/3
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Approximate degree

Let $f: D \to \{0,1\}$ where $D \subseteq \{0,1\}^n$.

A polynomial $p: \{0, 1\}^n \to \mathbb{R}$ is ε -approximation to f if

- $-\varepsilon \le p(x) \le 1 + \varepsilon$ for all $x \in \{0, 1\}^n$
- $|p(x) f(x)| \le \varepsilon$ for all $x \in D$

Approximate degree $deg_{\varepsilon}(f)$ of f is the least degree of a polynomial that ε -approximates f.

Default error $\varepsilon = \frac{1}{3}$

Corresponds to bounded error query model

Applications of deg

Upper bounds:

• Learning Algorithms

[Klivans-Servedio03, Klivans-Servedio06, Kalai-Klivans-Mansour-Servedio06]

• Approximate Inclusion-Exclusion

[Kahn-Linial-Samorodnitsky96, Sherstov08]

• Differentially Private Query Release

[Thaler-Ullman-Vadhan12, Chandrasekaran-Thaler-Ullman-Wan14]

 Formula & Graph Complexity Lower Bounds [Tal14,16ab]

Lower bounds:

Communication Complexity

[Sherstov07, Shi-Zhu07, Chattopadhyay-Ada08, Lee-Shraibman08,...]

• Circuit Complexity

[Minsky-Papert69, Beigel93, Sherstov08]

• Oracle Separations

[Beigel94, Bouland-Chen-Holden-Thaler-Vasudevan16]

• Secret Sharing Schemes

[Bogdanov-Ishai-Viola-Williamson16]

<u>Quantum query complexity</u>

Polynomial method

Acceptance probability of a randomized T-query algorithm is a polynomial of degree T.

The ε -approximate degree of a function is always at most its randomized query complexity with error ε .

Polynomial method

Acceptance probability of a quantum T-query algorithm is a polynomial of degree 2T.

The ε -approximate degree of a function is always at most $\frac{1}{2}$ its quantum query complexity with error ε .

Polynomial method

Acceptance probability of a quantum T-query algorithm is a polynomial of degree 2T.

The ε -approximate degree of a function is always at most ½ its quantum query complexity with error ε .

Advantages

- Robust.
 - Lifts to quantum communication [She11],
 - Can yield lower bounds against zero-, small-, and <u>unbounded-error</u> quantum algorithms [BBC+01, BCdWZ99],
 - Gives time-space tradeoffs [KSdW07]
- Transparent in identifying the hardness
 - Search vs decision problem
 - Can be more intuitive than some other methods

Other ways to prove lower bounds: "adversary" methods

- "Positive-weights" method [Ambainis02]
 - Easy to apply, but limited in power
- "Negative-weights" method [Høyer-Lee-Špalek07, ..., Reichardt11]
 - Tight characterization, but difficult to apply

Ordered search OS_{2^n}

Given: string $0^{x}1^{N-x}$, $N = 2^{n}$ Output: x

Hidden string

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Ordered search OS_{2^n}

Given: string $0^x 1^{N-x}$, $N = 2^n$ Output: parity(x)

Hidden string



Results: state-of-the-art

| | Ordered Search OS_{2^n} | Hidden String |
|---|--|--|
| Approximate degree, Quantum query complexity for decision problem | $O(n), \Omega(\sqrt{n})$ [BdW99] | O(n) [SS95][CIG+12] |
| Quantum query complexity for reconstruction problem | $\Theta(n)$ [BdW99, FGGS98, Amb99, HNS02, CL08] | $O(n)$ [SS95], $\Omega(n/\log^2 n)$ [CIG+12] |

Results: bounded error

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Results: unbounded error

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| Quantum query complexity for reconstruction problem Success pr γ | $\Theta(n\ -\lograc{1}{\gamma})$ [Implicit in Amb99] | $O\left(n - \log\frac{1}{\gamma}\right) \text{[SS95],}$ $\Omega(\gamma^2 \frac{n}{\log^2 n}) \text{[CIG+12]}$ This work: $\Omega(\frac{n}{\log^2 n} - \log\frac{1}{\gamma})$ |

• All results in this work were achieved using the same framework.

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EQ(a, b) = 1 iff a = b



Only need constant communication for constant error! Need $O(\log \log n)$ for $\frac{1}{\log n} = 2^{-\log \log n}$ error.


















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Exists a polynomial of degree 1: $parity(x) = \langle 1^n, x \rangle$

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Oracle structure



$$(Y(r, x))_i = \langle r_i, x \rangle$$

$$r = (r_1, r_2, \dots, r_m), r_i \in \{0, 1\}^n$$

$$r \leftarrow R$$
Random bit
Zero

Updated oracle structure R



t = poly(n)

Input: oracle access to Y(r, x)

1. Sample $j \leftarrow [t]$ u.a.r



...



Copy j

| Copy t | | | | | | | |
|--------|--|--|--|--|--|--|--|
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. . .



- 1. Sample $j \leftarrow [t]$ u.a.r
- 2. Simulate communication protocol using inner products from copy *j*





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Query algorithm for GT_i

Input: oracle access to Y(r, x)

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Query complexity $O(\log n \log \log n)$



Upper bound for GT_i

Claim 1'. With probability 2/3 over $r \leftarrow R$ u.a.r. for all $i \in \{0, 1\}^n$ there exists an algorithm A_i with oracle access to Y(r, x) that compute the corresponding GT_i with probability 2/3 over internal randomness and has query complexity at most $O(\log n \log \log n)$.

Acceptance probability of a randomized T-query algorithm is a polynomial of degree T.

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Lower bound for parity

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- Generalizes for any R with a "good" structure! $\Omega(\frac{n}{\log(size \ of \ oracle \ Y)})$
- Works for approximation to any error

Lower bound for parity

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- **Proof idea 1:** We consider polynomials over $\{-1, 1\}$ instead of $\{0, 1\}$
- Proof idea 2: All monomials of degree < n are orthogonal to parity in {−1, 1} basis.
- **Proof idea 3:** Getting a monomial of x of degree n by multiplying $\frac{n}{\log n}$ bits of Y(r, x) is improbable.

Lower bound for parity: key lemma



Final result for OS

Theorem. Every polynomial that approximates OS_{2^n} requires degree $\Omega(n/\log^2 n)$.



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- New (and almost tight) lower bound for approximate degree of Ordered search: $\Omega(n/\log^2 n)$
- Generalizable to unbounded error regime: $\Omega(\frac{n}{\log^2 n} \log \frac{1}{\gamma})$
- Same lower bounds for Hidden string problem using the same approach
- As a corollary, lower bounds on quantum query complexity of decision versions
- New framework for lower bounds on oracle identification problems

Open problems:

- Using the easiness of one problem in the presence of additional information to prove the hardness of another
- Using this framework for other open problems
- Using this framework in other settings: circuit complexity, proof complexity, massive parallel computation model, ...
- Closing the gap for ordered search and hidden string: $\Omega(n/\log^2 n)$ and O(n)

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