Maximum entanglement of mixed symmetric states under unitary transformations

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Motivation

The problem studied by Verstraete, Audenaert and De Moor in [1] – about which global unitary operations maximize the entanglement of a bipartite qubit system – is revisited, extended and solved when permutation symmetry on the qubits is imposed [2]. This condition appears naturally in bosonic systems or spin systems [3]. We fully characterize the set of absolutely separable symmetric states (SAS) for two qubits and provide fairly tight bounds for three qubits. In particular, we find the maximal radius of a ball of SAS states around the maximally mixed state in the symmetric sector, and the minimum radius of a ball that includes the set of SAS states, for both two and three qubits.

Useful concepts

The **negativity** $\mathcal{N}$ of a state $\rho \in \mathcal{B}(\mathcal{H})$, defined in terms of the negative eigenvalues $\Lambda_k < 0$ of the partial transpose of ρ

$$\mathcal{N}(\rho) = -2 \sum_k \Lambda_k \,, \tag{1}$$

is a measure of entanglement for qubit-qubit and qubit-qutrit systems [4]. The maximum entanglement in the $SU(4)$-orbit of a 2-qubit state is [1]

$$\max_{U \in SU(4)} \mathcal{N}(U\rho U^\dagger) = \max\left(0, \sqrt{(\lambda_1-\lambda_3)^2 + (\lambda_2-\lambda_4)^2} - \lambda_2 - \lambda_4\right), \tag{2}$$

where $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \lambda_4$ is the eigenspectrum of ρ. In particular, Eq. (2) characterizes the set of **absolutely separable states** which are the states that remain separable after any global unitary transformation [5].

Problem statement

When there is a permutation invariance restriction on the quantum states, this reduces their allowed eigenspectrum and the admissible global unitary transformations. In this case, **what is the maximum entanglement achievable under a global unitary transformation in the symmetric subspace ?**

Qubit-qubit system $\mathcal{H}_2^{\otimes 2}$	Symmetric 2-qubit system $\mathcal{H}_2^{\vee 2}$
ρ−spectrum: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$	ρ_S−spectrum: $(\tau_1, \tau_2, \tau_3, 0)$
$\max_{U\in SU(4)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S\in SU(3)} \mathcal{N}\left(U_S\rho_S U_S^\dagger\right)$
Qubit-qutrit system $\mathcal{H}_2 \otimes \mathcal{H}_3$	**Symmetric 3-qubit system $\mathcal{H}_2^{\vee 3}$**
ρ−spectrum: $(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)$	ρ_S−spectrum: $(\tau_1, \tau_2, \tau_3, \tau_4, 0, 0)$
$\max_{U\in SU(6)} \mathcal{N}\left(U\rho U^\dagger\right)$	$\max_{U_S\in SU(4)} \mathcal{N}\left(U_S\rho_S U_S^\dagger\right)$

We define a **Symmetric Absolutely Separable** (SAS) state ρ_S as a state verifying $\mathcal{N}(U_S\rho_S U_S^\dagger) = 0$ for any unitary U_S leaving the symmetric subspace invariant. There are two balls centred on the maximally mixed state in the symmetric subspace ρ_0 which qualitatively describe the extension of the set $\mathcal{A}_{\text{sym}}$ of SAS states in $\mathcal{B}(\mathcal{H})$, with radii (see Fig. 1):

$R_{SAS} \equiv$ minimal radius of the ball centred on ρ_0 and containing $\mathcal{A}_{\text{sym}}$,
$r_{SAS} \equiv$ maximal radius of the ball centred on ρ_0 and contained in $\mathcal{A}_{\text{sym}}$.

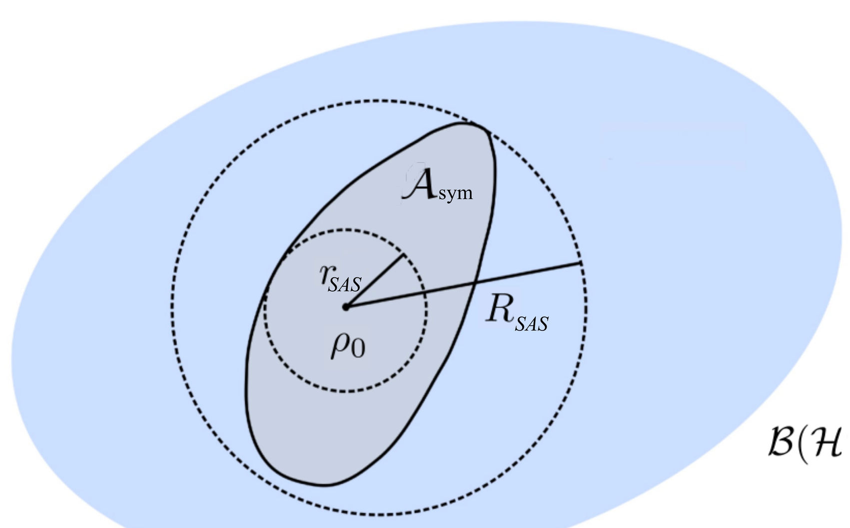

Fig. 1: The set $\mathcal{A}_{\text{sym}}$ of symmetric absolutely separable (SAS) states.

Symmetric 2-qubit states

Theorem 1 *Let $\rho_S \in \mathcal{B}(\mathcal{H}_2^{\vee 2})$ with spectrum $\tau_1 \geq \tau_2 \geq \tau_3$. It holds that*

$$\max_{U_S\in SU(3)} \mathcal{N}\left(U_S\rho_S U_S^\dagger\right) = \max\left\{0, \sqrt{\tau_1^2 + (\tau_2-\tau_3)^2} - \tau_2 - \tau_3\right\}. \tag{3}$$

Corollary 1 $\rho_S \in \mathcal{A}_{sym} \;\Leftrightarrow\; \sqrt{\tau_2} + \sqrt{\tau_3} \geq 1$.

From Corollary 1, we find for r_{SAS} and R_{SAS} (see Figs. 1 and 2)

$$\boxed{r_{SAS} = 1/2\sqrt{6}} \qquad \boxed{R_{SAS} = 2/3\sqrt{6}} \tag{4}$$

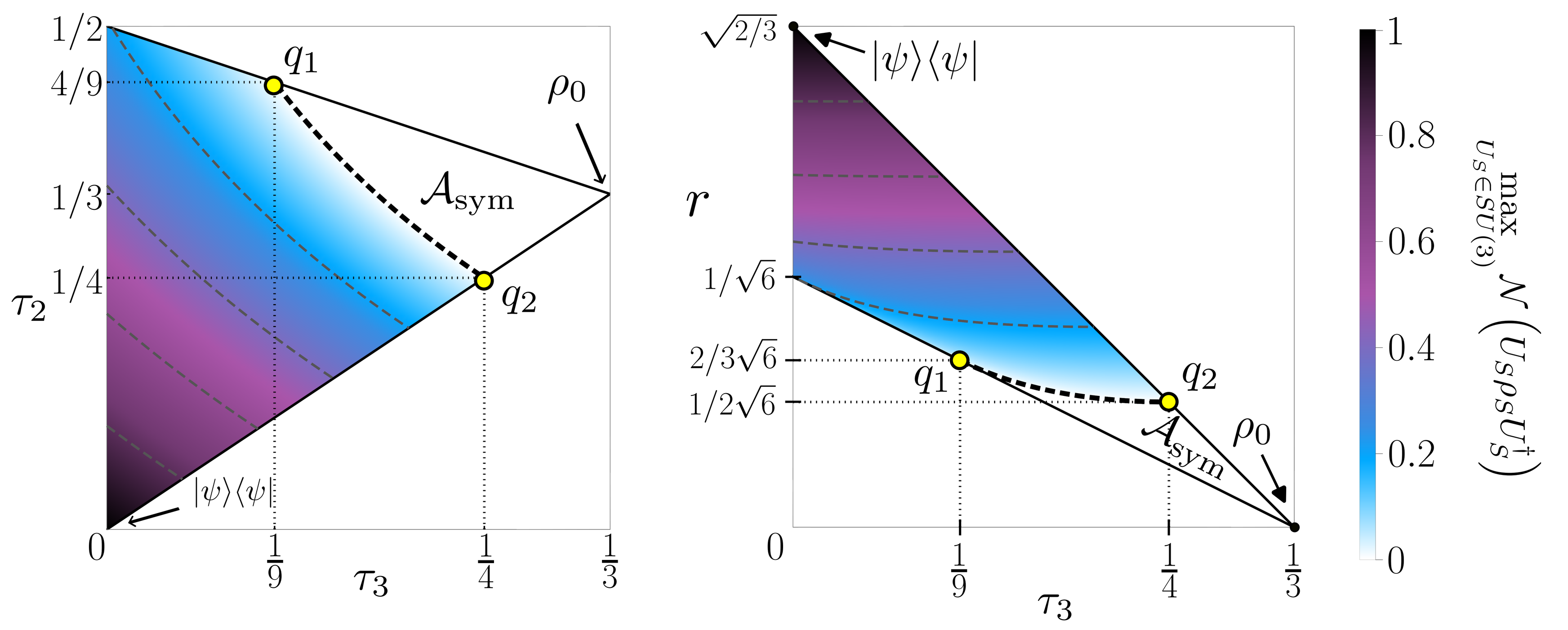

Fig. 2: Density plot of the maximum negativity (3) attained in the $SU(3)$ orbit of $\rho_S \in \mathcal{B}\left(\mathcal{H}_2^{\vee 2}\right)$ over the (τ_3, τ_2) plane (left) and the (r, τ_3) plane (right). The grey dashed lines are contour curves of constant negativity. The set $\mathcal{A}_{\text{sym}}$ is depicted by the white region bounded on the left by the black dashed curve.

Symmetric 3-qubit states

Observation 1 *A symmetric three-qubit state ρ_S cannot be SAS if its eigenspectrum $\tau_1 \geqslant \tau_2 \geqslant \tau_3 \geqslant \tau_4$ satisfies*

$$\tau_1 > \sqrt{3\,\tau_3\tau_4} \quad \wedge \quad (3\tau_1 - 2\tau_2)^2\tau_3 + 3(\tau_2^2 - \tau_3^2)\tau_4 > 9\tau_3\tau_4^2\,. \tag{5}$$

The previous result is an effective non-SAS witness because the only undetected non-SAS states all have a negativity of order 10^{-4} at most. Numerical calculations suggest that Obs. 1 gives the proper radii r_{SAS} and R_{SAS} (equality reached in the yellow points p_1 and p_4 of Fig. 3)

$$\boxed{r_{SAS} \leq 1/2\sqrt{19}} \qquad \boxed{R_{SAS} \leq \sqrt{3}/10}$$

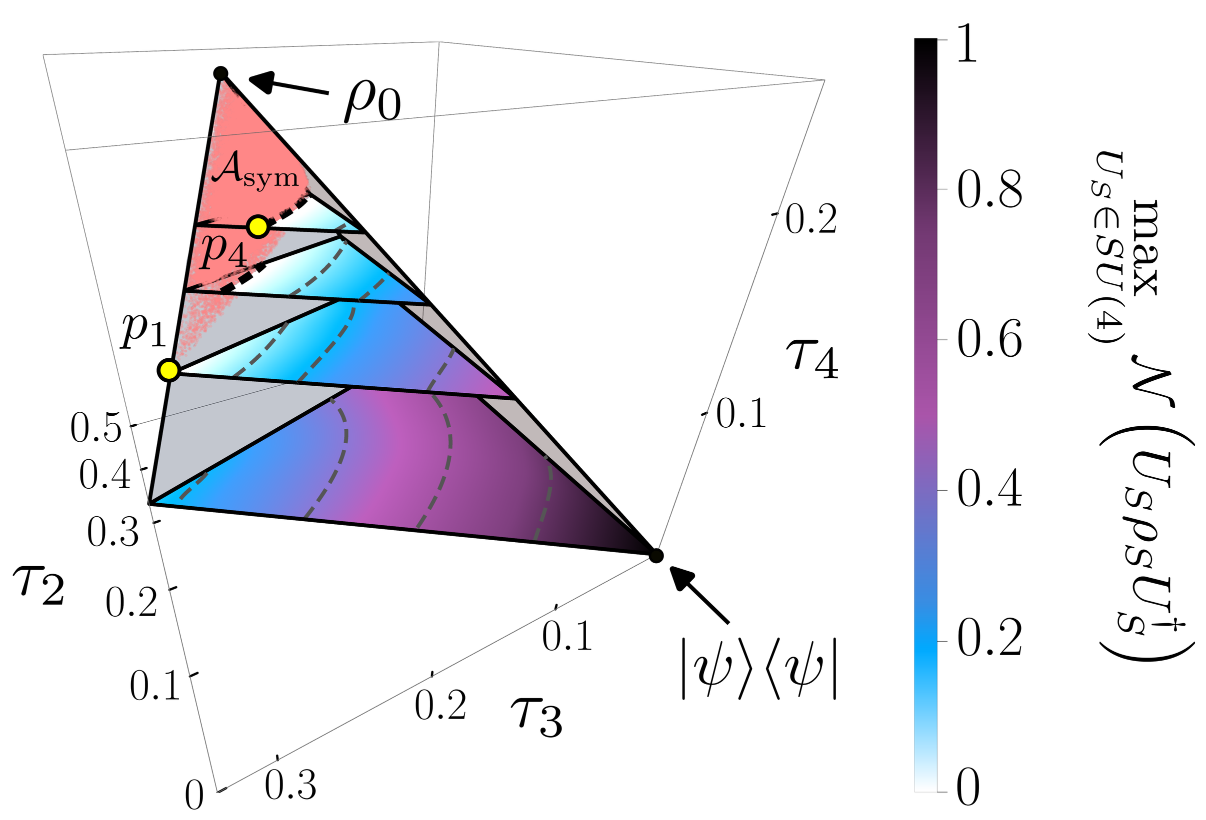

Fig. 3: Maximum negativity in the $SU(4)$-orbit of $\rho_S \in \mathcal{B}\left(\mathcal{H}_2^{\vee 3}\right)$ in the (τ_2, τ_3, τ_4) simplex for $\tau_4 = 0, 1/10, 3/20, 7/38$. The pink points are SAS states calculated numerically. See Ref. [2] for more details. The contour curves denote where the maximum negativity is equal to 0.8, 0.6, 0.4, 0.2, 0.1, respectively. The yellow points p_1 and p_4 correspond to the eigenvalues $(\tau_1, \tau_2, \tau_3, \tau_4) = (3,\ 3,\ 3,\ 1)/10$ and $(13, 9, 9, 7)/38$.

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