Improving social welfare in non-cooperative games with different types of quantum resources

Alastair A. Abbott[1], Mehdi Mhalla[2], and Pierre Pocreau[2]

[1]Univ. Grenoble Alpes, Inria, 38000 Grenoble, France

[2]Univ. Grenoble Alpes, CNRS, Grenoble INP, LIG, 38000 Grenoble, France

Summary

- Non-cooperative games involving multiple players exhibit equilibria wherein no player has an incentive to deviate from their strategy
- The quality of an equilibrium can be quantified by its social welfare – the mean payoff each player receives
- Access to shared quantum resources may allow better cooperation, and hence better equilibria
- We consider two scenarios: in one, players may make measurements directly on a quantum state, while in the other, they delegate the measurement to a referee
- We compare the classes of equilibria obtainable in each setting as well as their maximal social welfare as a function of the bias of the game

Non-cooperative games

A non-cooperative game G between n players is defined by:

- A set of **questions** $T \subseteq \{0,1\}^n$
- A **prior distribution** Π over the questions T
- A set of **valid answers** $A = \{0,1\}^n$
- A **payoff function** u_i for each player i, with $u_i(a,t) \in \mathbb{R}$.
  - We consider payoff functions with the form

$$u_i(a,t) = \begin{cases} 0 \text{ if } (a,t) \notin \mathcal{W} \\ v_0 \text{ if } a_i = 0 \text{ and } (a,t) \in \mathcal{W} \\ v_1 \text{ if } a_i = 1 \text{ and } (a,t) \in \mathcal{W}, \end{cases}$$

  with $v_0, v_1 > 0$ and $\mathcal{W} \subseteq A \times T$ a set of "winning input-output pairs"
  - Ratio v_0/v_1 controls the bias of game

Example: Winning conditions for two 5-player games: $\mathrm{NC}_{00}(C_5)$ and $\mathrm{NC}_{01}(C_5)$ [1]

Question $t_1t_1t_2t_3t_5$	Winning condition, $\mathrm{NC}_{00}(C_5)$
10000	$a_4 \oplus a_0 \oplus a_1 = 0$
01000	$a_0 \oplus a_1 \oplus a_2 = 0$
00100	$a_1 \oplus a_2 \oplus a_3 = 0$
00010	$a_2 \oplus a_3 \oplus a_4 = 0$
00001	$a_3 \oplus a_4 \oplus a_0 = 0$
11111	$a_0 \oplus a_1 \oplus a_2 \oplus a_3 \oplus a_4 = 1$

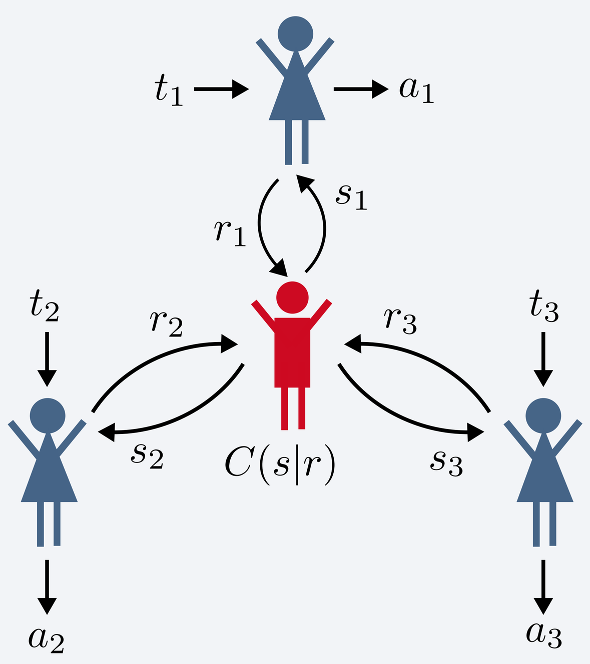
Question $t_1t_1t_2t_3t_5$	Winning condition, $\mathrm{NC}_{01}(C_5)$
10100	$a_4 \oplus a_0 \oplus a_1 = 0$
01010	$a_0 \oplus a_1 \oplus a_2 = 0$
00101	$a_1 \oplus a_2 \oplus a_3 = 0$
10010	$a_2 \oplus a_3 \oplus a_4 = 0$
01001	$a_3 \oplus a_4 \oplus a_0 = 0$
11111	$a_0 \oplus a_1 \oplus a_2 \oplus a_3 \oplus a_4 = 1$

Solutions and equilibria

- Each player follows a local strategy to produce their answer
- In general, they may also have access to a shared correlation in the form of an advice s_i provided by a mediator with probability $C(s_1 \dots s_n | r_1 \dots r_n)$
- A solution (set of strategies for each player, defined by functions f_i and g_i) induces a distribution

$$P(a|t) = \sum_\lambda \Lambda(\lambda) \sum_{s\,:\,\forall i,\, g_i(t_i,s_i,\lambda_i)=a_i} C(s_1 \dots s_n | f(t_1,\lambda_1) \dots f(t_n,\lambda_n))$$

  - We can generally consider just deterministic strategies



- A solution is a **Nash equilibrium** if no player can increase their mean payoff by changing their strategy: $\forall i\, \forall t_i, r_i \in T_i\, \forall \mu_i : T_i \times A_i \to A_i$,

$$\sum_{t_{-i},a_{-i}} u_i(a,t) P(a|t)\Pi(t) \geq \sum_{t_{-i},a_{-i}} u_i(\mu_i(t_i,a_i)a_{-i},t) P(a|r_i t_{-i})\Pi(t)$$

  - Nash equilibria play important roles in applications from economics to engineering
  - Different correlations C lead to different equilibria: Nash (no correlation), Corr (shared randomness), B.I. (belief invariant, or no-signalling), ...
- For a game G, the **social welfare** of a solution is

$$SW_G(P) = \frac{1}{n} \sum_i \sum_{a,t} u_i(a,t) P(a|t) \Pi(t).$$

Two types of quantum advices

Question: How can quantum resources lead to new equilibria or improve social welfare?

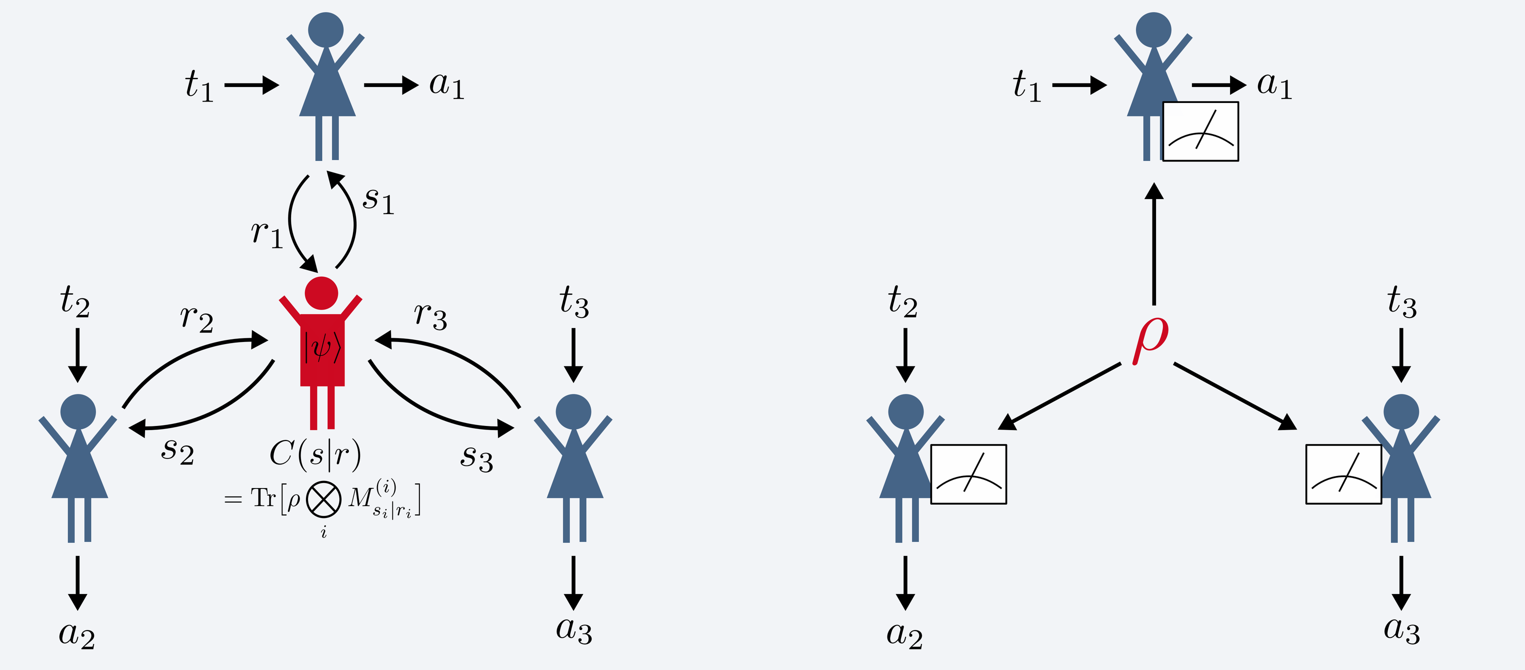
We identify two types of quantum advice leading to different equilibria:

- Quantum correlated advice: Advice C is obtained from measurements on a quantum system:

$$C(s|r) = \mathrm{Tr}\left[\rho(M^{(1)}_{s_1|r_1} \otimes \cdots \otimes M^{(n)}_{s_n|r_n})\right]$$

  - Measurement delegated to mediator, or performed by parties with quantum "black-boxes"
- Shared quantum state [2]: Each player measures a shared quantum state to determine their output a_i
  - Direct access to quantum resource
  - Notion of equilibria modified: a player can deviate by choosing any other local POVM: $\forall i\, \forall t_i\, \forall N^{(i)} = \{N^{(i)}_{a_i|r_i}\}_{r_i}$

$$\sum_{t_{-i},a} u_i(a,t)\,\mathrm{Tr}\left(\rho \cdot \bigotimes_j M^{(j)}_{a_j|t_j}\right)\Pi(t) \geq \sum_{t_{-i},a} u_i(a,t)\,\mathrm{Tr}\left(\rho \cdot \bigotimes_{j\neq i} M^{(j)}_{a_j|t_j} \otimes N^{(i)}_{a_i|t_i}\right)\Pi(t)$$



Classical access – $Q_{\mathrm{corr}}(G)$ Quantum access – $Q(G)$

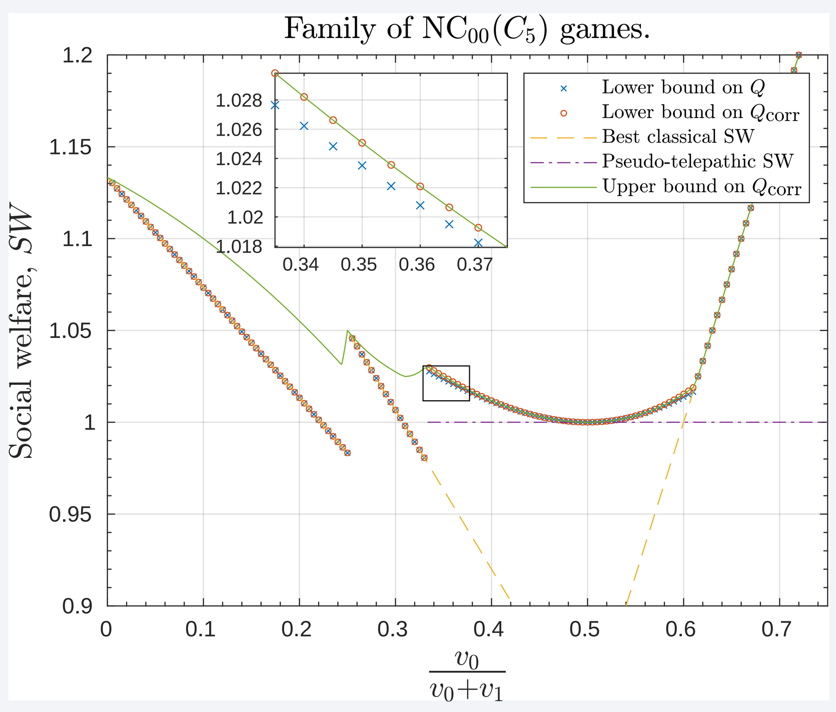
- For any game G, the sets of equilibria satisfy

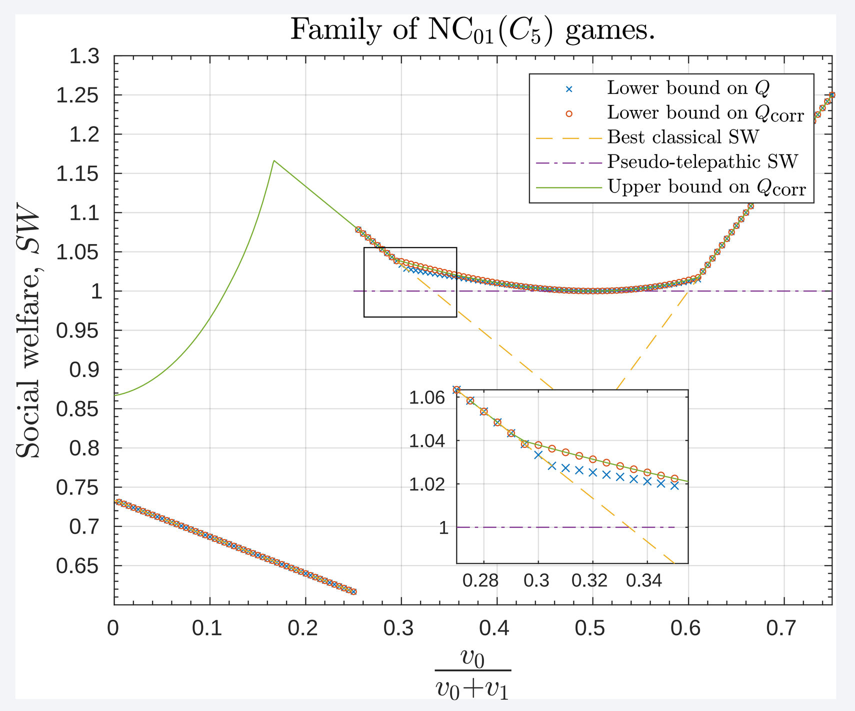
$$\mathrm{Nash}(G) \subset \mathrm{Corr}(G) \subset Q(G) \subset Q_{\mathrm{corr}}(G) \subset \mathrm{B.I.}(G) \subset \mathrm{Comm}(G)$$

- We show using self-testing methods [3] that for some game G, $Q(G) \subsetneq Q_{\mathrm{corr}}(G)$
  - Players who delegate quantum measurements can reach more equilibria!

Results: Social welfare of different solutions

- We optimised the social welfare over different solutions classes for three types of games: $\mathrm{NC}_{00}(C_5)$, $\mathrm{NC}_{01}(C_5)$, and $\mathrm{NC}(C_3)$ (not shown here) [1]
  - **Best classical SW:** computed exactly
  - **Graph state SW:** pseudo-telepathic equilibria using GHZ states [1]
  - **Seesaw lower bound:** numerical optimisation by iterating SDPs to find explicit solutiobs lower-bounding QSW over $Q(G)$ and $Q_{\mathrm{corr}}(G)$
  - **NPA upper bound:** SDP hierarchy providing dimension-independent upper bound on equilibria in $Q_{\mathrm{corr}}(G)$ [4]





Conclusions and open questions

- Two different ways to use quantum resources lead to distinct classes of equilibria
- A strict separation between $Q(G)$ and $Q_{\mathrm{corr}}(G)$
- Quantum social welfare can be improved beyond pseudo-telepathic solution
- Method to directly obtain upper bounds on $Q(G)$?

References and acknowledgements

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