Bipartite entanglement and its detection by local generalized measurements

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Motivation: Entanglement detection with local measurements

- Entanglement detection with local measurements can be performed over far distant distances
- Relevant for quantum key distribution and quantum communication
- Consider a bipartite system with dimensions d_A and d_B and observers Alice and Bob
- Comparison of the success rate for different type of informationally complete measurements
  - Local orthogonal hermitian operator basis (LOO)
  - (N, M)-POVMs
- Sufficient condition involving trace norms are examined for arbitrary states

Hermitian operator bases

- Basis of hermitian $d \times d$ matrices
- The set of hermitian operators $G = (G_1, \cdots, G_{d^2})^T, G_i^\dagger = G_i$
- Orthogonal to the Hilbert-Schmidt scalar product $\mathrm{Tr}\{G_i G_j\} = \delta_{ij}$
- A special basis is $\tilde{G} = (\tilde{G}_1 = \mathbb{1}_d/\sqrt{d}, \tilde{G}_2 \cdots, \tilde{G}_{d^2})^T$ and $\mathrm{Tr}\{G_i\} = 0$ for $i \in 2, \ldots, d^2$

(N, M)-POVMs

Definition [1]

- Consider a qudit of dimension d
- A tuple of $\Pi = \{\Pi_1, \ldots, \Pi_{NM}\}$ positive operators consisting of N POVMs with M results
- The completeness relation of a single POVM $\sum_{a=1}^M \Pi_{i(\alpha,a)} = \mathbb{1}_d$
- The (N, M)-POVMs fulfill for $\alpha \neq \beta$

$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\} = \frac{d}{M},$$
$$\mathrm{Tr}\{\Pi_{i(\alpha,a)}\,\Pi_{i(\alpha,a')}\} = x\,\delta_{a,a'} + (1-\delta_{a,a'})\frac{d-Mx}{M(M-1)},$$
$$\mathrm{Tr}\{\Pi_{\alpha,a}\,\Pi_{\beta,b}\} = \frac{d}{M^2}$$

- The parameter is limited by $d/M^2 < x \leq \min(d^2/M^2, d/M)$
- Informationally complete for $(M-1)N + 1 = d^2$
- Examples
  - Mutually unbiased measurements $M = d$
  - Mutually unbiased bases $M = d$ and $x = 1$
  - GSIC $M = d^2$
  - SIC-POVM $M = d^2$ and $x = 1/d^2$

Properties [2, 3]

- Basis expansion $\Pi = G^T S$
- Transformation properties with $\Gamma = \frac{xM^2-d}{M(M-1)}$

$$\left(S^T S\right)_{i(\alpha,a),j(\alpha',a')} = \Gamma\delta_{i,j} - \frac{\Gamma}{M}\delta_{\alpha,\alpha'} + \frac{d}{M^2}$$

- Spectrum Sp$(S^T S) = \{\Gamma^{(N(M-1))}, \frac{dN}{M}^{(1)}, 0^{(N-1)}\}$
- The matrix SS^T is diagonal with Sp$(S^T S) = \{\Gamma^{(N(M-1))}, \frac{dN}{M}^{(1)}\}$

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CROSSING

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Entanglement detection with trace norms

Correlation matrices

- Consider arbitrary sets of hermitian operators $\mathcal{A} = \{A_i; i = 1, \cdots, N_A\}$ and $\mathcal{B} = \{B_j; j = 1, \cdots, N_B\}$ of the local observes Alice and Bob
- Definition of the correlation matrix

$$(C(\mathcal{A}, \mathcal{B}|\varrho))_{ij} = \mathrm{Tr}\{A_i \otimes B_j \left(\varrho - \varrho^A \otimes \varrho^B\right)\}$$

- Separable states fulfill

$$||C(\mathcal{A}, \mathcal{B}|\varrho)||_1^2 \leq \Sigma_A \Sigma_B$$
$$\Sigma_A = \max_{\sigma^A} \sum_{i=1}^{N_A} \left(\mathrm{Tr}\{A_i \sigma^A\}\right)^2 - \left(\mathrm{Tr}\{A_i \varrho^A\}\right)^2,$$
$$\Sigma_B = \max_{\sigma^B} \sum_{j=1}^{N_B} \left(\mathrm{Tr}\{B_j \sigma^B\}\right)^2 - \left(\mathrm{Tr}\{B_j \varrho^B\}\right)^2$$

- LOOs $G^i = \{G_1, \ldots, G_{d_i^2}\}$ [4]

$$||C(G^A, G^B|\varrho)||_1^2 \leq (1 - \mathrm{Tr}\{(\varrho^A)^2\})(1 - \mathrm{Tr}\{(\varrho^B)^2\})$$

- For (N, M)-POVMs the scaling relation holds

$$||(S^A)^T C(G^A, G^B|\varrho) S^B||_1^2 = \Gamma_A \Gamma_B ||C(G^A, G^B|\varrho)||_1^2$$

- The entanglement detection with (N, M)-POVMs

$$||C(\Pi^A, \Pi^B|\varrho)||_1^2 \leq \Gamma_A \Gamma_B (1 - \mathrm{Tr}\{(\varrho^A)^2\})(1 - \mathrm{Tr}\{(\varrho^B)^2\})$$

$\Longrightarrow$ **(N, M)-POVMs and LOOs are leading to the same inequalities for entanglement detection [2]**

Joint probabilities [1]

- Definition of the joint probability matrix for two local (N_i, M_i)-POVMs

$$P(\Pi^A, \Pi^B)|\varrho) = \mathrm{Tr}\{(\Pi^A)^T \otimes \Pi^B \varrho\}$$

- Separable states fulfill

$$||P(\Pi^A, \Pi^B|\varrho)||_1^2 \leq \left(\Gamma_A \left(1 - \frac{1}{d_A}\right) + \frac{N_A}{M_A}\right)\left(\Gamma_B \left(1 - \frac{1}{d_B}\right) + \frac{N_B}{M_B}\right)$$

- Expansion into LOOs [2]

$$||P(\Pi^A, \Pi^B|\varrho)||_1 = ||\sqrt{\Lambda^A} P(\tilde{G}^A, \tilde{G}^B \varrho)\sqrt{\Lambda^B}||_1$$

with the diagonal matrices of dimension $d_A \times d_A$ and $d_B \times d_B$

$$(\Lambda^A)_{11} = \gamma_A(d_A + 1),$$
$$(\Lambda^A)_{\nu\nu} = \Gamma_A = \gamma_A \frac{d_A \tilde{x}_A - 1}{d_A - 1},$$
$$(\Lambda^B)_{11} = \gamma_B(d_B + 1),$$
$$(\Lambda^B)_{\mu\mu} = \Gamma_B = \gamma_B \frac{d_B \tilde{x}_B - 1}{d_B - 1}$$

To compare different types of (N, M)-POVMs the rescaled parameters are introduced

$$\tilde{x}_A = \frac{x_A M_A^2}{d_A^2}, \quad \tilde{x}_B = \frac{x_B M_B^2}{d_B^2}.$$

and the separable states fulfill

$$\left|\left|\sqrt{\frac{\Lambda^A}{\gamma_A}} P(\tilde{G}^A, \tilde{G}^B|\varrho)\sqrt{\frac{\Lambda^B}{\gamma_B}}\right|\right|_1 \leq \sqrt{1 + \tilde{x}_A}\sqrt{1 + \tilde{x}_B}$$

$\Longrightarrow$ **Entanglement detection only depends on the rescaled parameters of the (N, M)-POVMs**

$\Longrightarrow$ MUBs and SICs are able to detect the same entangled states

$\Longrightarrow$ Entanglement detection with correlation matrices can detected more entangled states than joint probabilities

References

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Hit and run Monte-Carlo method

- Sufficient conditions for entanglement detection are tested on bipartite qudit systems of dimension $2 \leq d_i \leq 4$
- A two qudit density matrix can be represent by

$$\varrho = \frac{\mathbb{1}_{d_A d_B}}{d_A d_B} + \sum_{i=2}^{d_A^2} a_i \tilde{G}_i^A \otimes \tilde{G}_1^B + \sum_{j=2}^{d_B^2} b_j \tilde{G}_1^A \otimes \tilde{G}_j^B + \sum_{i=2}^{d_A^2}\sum_{j=2}^{d_B^2} t_{ij} \tilde{G}_i^A \otimes \tilde{G}_j^B$$

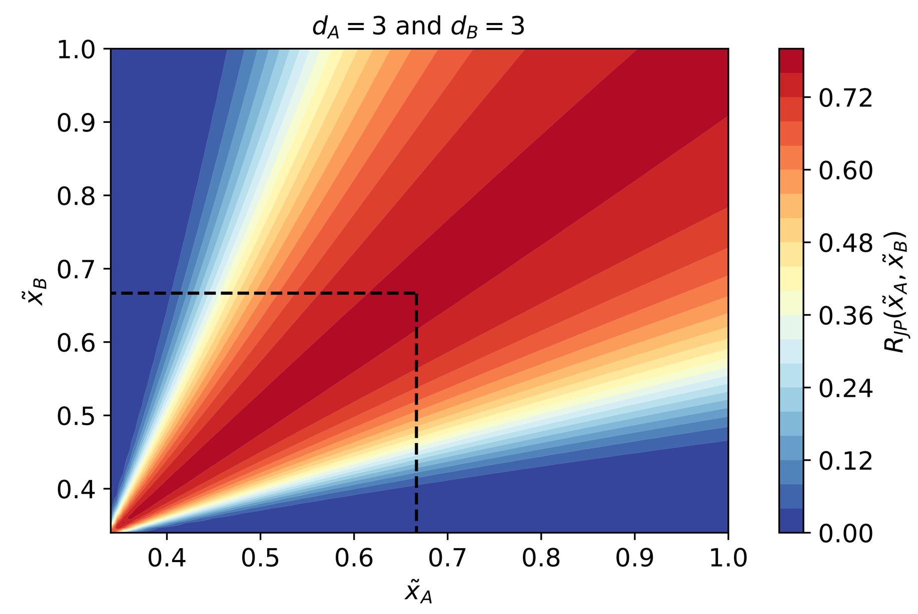
- The density matrices are subset of a real vector space of dimension $d_A^2 d_B^2 - 1$,
- A random walk is performed in the real vector space, but only the points are contained which lead to a positive density matrix [5]
- This method was able to generate 10^8 states

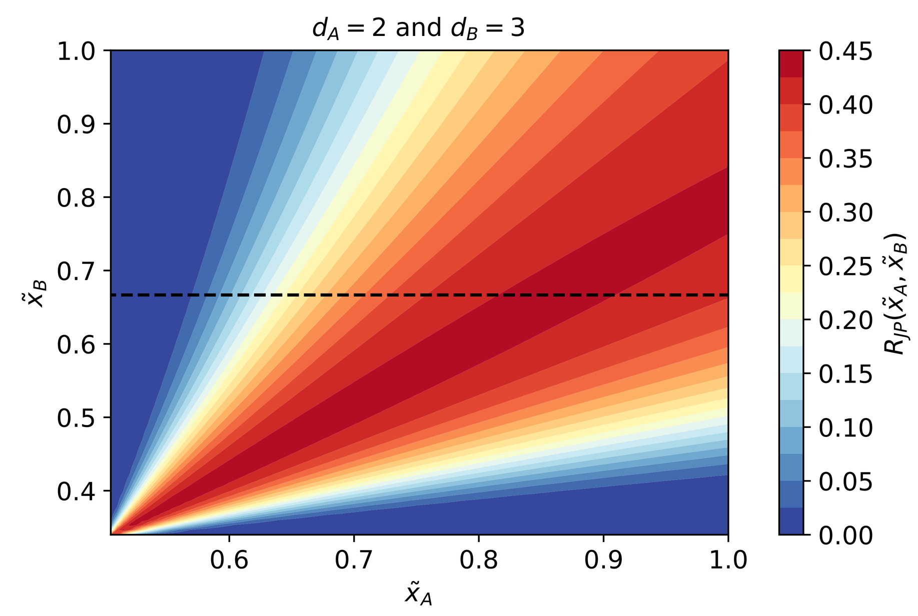
Simulation result for qudit systems

- The Euclidean volume ratios R of the entangled states over all states are calculated and compared with entangled states detected py Peres-Horodecki-condition (PPT) [6]

(d_A, d_B)	R_{PPT}	$R_{JP,SIC}$	$R_{Cor,LOO}$
$(2,2)$	$0,75784$ $\pm 1,7(4)$	$0,67060$ $\pm 2,2(4)$	$0,68860$ $\pm 2,1(4)$
$(2,3)$	$0,97303$ $\pm 7(5)$	$0,39732$ $\pm 5,6(4)$	$0,43853$ $\pm 5,5(4)$
$(2,4)$	$0,998696$ $\pm 1,6(5)$	$0,02710$ $\pm 2,7(4)$	$0,04504$ $\pm 3,5(4)$
$(3,3)$	$0,999895$ $\pm 4(6)$	$0,75680$ $\pm 8,2(4)$	$0,76364$ $\pm 8,1(4)$
$(3,4)$	1 ± 0	$0,3605$ $\pm 1,8(3)$	$0,3795$ $\pm 1,8(3)$
$(4,4)$	1 ± 0	$0,6378$ $\pm 7,7(3)$	$0,6419$ $\pm 7,7(3)$

- The dependence of the volume ratios calculated with joint probabilities for the parameters x_A and x_B





Conclusion and Outlook

- The symmetry of (N, M)-POVMs shows characteristic scaling properties for entanglement detection
- Entanglement detection with joint probabilities only depends on the rescaled parameters $\tilde{x}_A$ and $\tilde{x}_B$ and is independent for correlation matrices
- The scaling relation can be investigated for multipartite systems