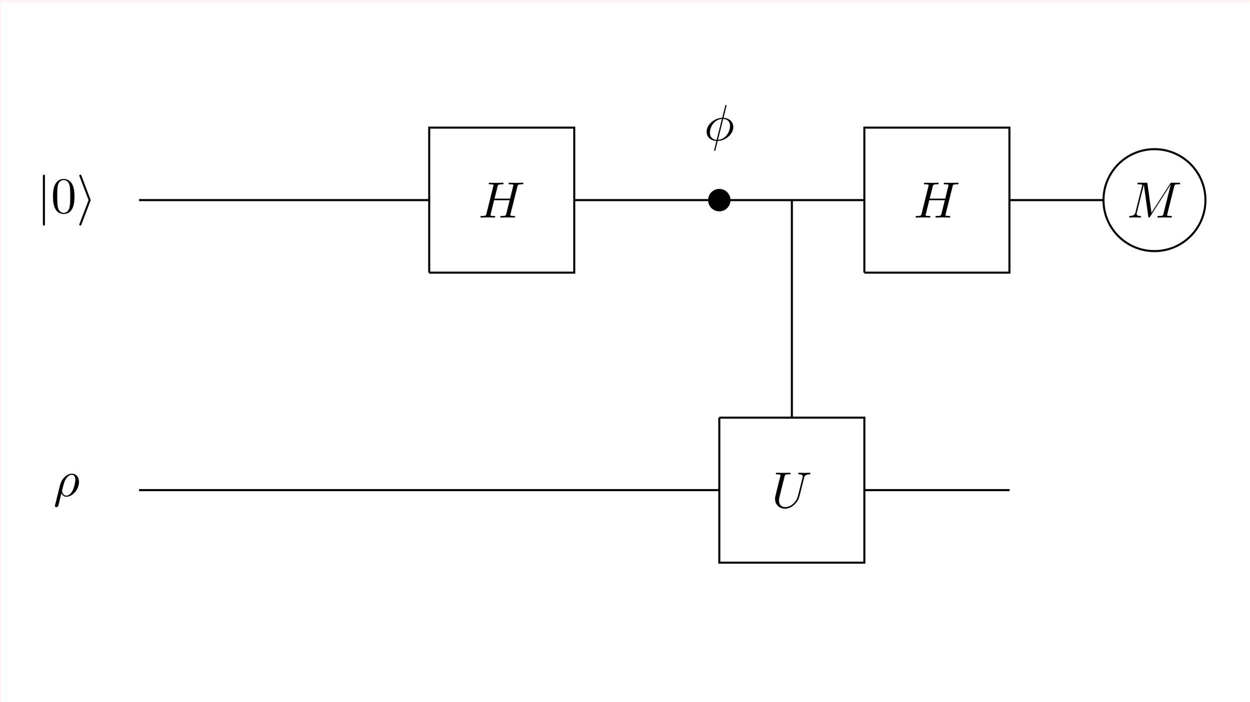
Entanglement Meter: Estimation of entanglement with single copy in Interferometer

Som Kanjilal, Vivek Pandey and Arun Kumar Pati

Harish-Chandra Research Institute, HBNI

Abstract: **Efficient certification and quantification of high dimensional entanglement of composite systems are challenging both theoretically as well as experimentally. Here, we demonstrate how to measure the linear entropy, negativity and the Schmidt number of bipartite systems from the visibility of Mach–Zehnder interferometer using single copies of the quantum state. We also propose how to measure the mutual predictability experimentally from the intensity patterns of the interferometric set-up without having to resort to local measurements of mutually unbiased bases. Furthermore, we show that the entanglement witness operator can be measured in a interference setup and the phase shift is sensitive to the separable or entangled nature of the state.**

Quantum properties from interferometric visibility



- If we detect the intensity corresponding to the interferometric arm $|0\rangle$ it results in

$$\begin{aligned}\mathcal{I} &= \mathrm{tr}\left((|0\rangle\langle 0|\otimes\mathbb{I})\,\rho_f\right)\\ &= \frac{1}{4}[1+|\mathrm{tr}(U\rho)|\cos(\phi-\arg[\mathrm{tr}(U\rho)])],\end{aligned} \tag{1}$$

  where $\mathrm{tr}(U\rho) = |\mathrm{tr}(U\rho)|\exp[i\arg[\mathrm{tr}(U\rho)]]$. It is seen that the action of the controlled-unitary on ρ modifies the visibility of the interference pattern by the factor $V = |\mathrm{tr}(U\rho)|$ and the phase is modified by the argument of $\mathrm{tr}(U\rho)$, i.e., $\alpha = \arg[\mathrm{tr}(U\rho)]$ for an input state ρ undergoing unitary evolution.
- Thus, we can infer the quantity $\mathrm{tr}(U\rho)$ by measuring the change in the visibility and the phase shift. We will show that we can choose the unitary U suitably, such that, from the quantity $\mathrm{tr}(U\rho)$, we can determine whether the state is entangled and the amount of entanglement present in the bipartite system.

Linear entropy with a single copy in Interferometer

- We show that a single copy of the input state (or even a subsystem of the input state) is sufficient to measure the linear entropy using the notion of quantum oracle (a quantum process required to solve a task).
- In particular, given single copy of $|\Psi\rangle_{AB}$ an oracle implements the unitary $U = (I-2P)$, where $P = |\Psi\rangle\langle\Psi|_{AB}$. We send one of the local subsystem ρ_A and another system which is prepared in a maximally mixed state $\frac{I_B}{d}$, where $d = \dim(\mathcal{H}_B)$. We then have

$$|tr(U\rho_{AB})| = 1 - \frac{2}{d}tr(\rho_A^2).$$

- For a bipartite mixed state ρ_{AB} with decomposition $\{p_i, |\Psi_i\rangle\}$, the convex roof extended linear entropy is defined as

$$\mathcal{E}_c(\rho_{AB}) = \min_{\{p_i,|\Psi_i\rangle\}} \left(p_i\mathcal{E}(|\Psi_i\rangle)\right), \tag{2}$$

  here $\mathcal{E}(|\Psi_i\rangle) = \left(1-\mathrm{tr}(\rho_{Ai}^2)\right)$ is the linear entropy of $|\Psi_i\rangle$, and $\rho_{Ai} = \mathrm{tr}_B(|\Psi_i\rangle\langle\Psi_i|)$.
- From the definition of convex roof extended linear entropy one can then obtain

$$\mathcal{E}_c(\rho_{AB}) \le 1 - \frac{d(1-V)}{2}. \tag{3}$$

Measuring Schmidt number with a single copy

- Let Schmidt rank of a pure state $|\Psi\rangle$ is denoted as $SR(|\Psi\rangle)$. The Schmidt number of a mixed state is defined as

$$SN(\rho) = \min_{\{p_i,|\Psi_i\rangle\},\rho=\sum_i p_i|\Psi_i\rangle\langle\Psi_i|} \max_i SR(|\Psi_i\rangle) \tag{4}$$

- Let us denote the set of all states with Schmidt number less than of equal to k as S_k i.e., $S_k := \{\rho : SN(\rho) \le k\}$. For a state $\rho \in S_k$, we have the following

$$\max_{|\Psi\rangle\in S_B}\langle\Psi|\,\rho\,|\Psi\rangle \le \frac{k}{d}, \tag{5}$$

  S_B is the set of maximally entangled states.
- Consider that a bipartite state ρ_{AB} and unknown Schmidt number k is used as input of the interferometric setup. The oracle produces an unitary $U_* = I - 2|\Psi_*\rangle\langle\Psi_*|$, where $\max_{|\Psi\rangle\in S_B}\langle\Psi|\,\rho\,|\Psi\rangle = \langle\Psi_*|\,\rho\,|\Psi_*\rangle$, for $\rho_{AB} \in S_k$. We then obtain

$$0 \le \frac{1\pm V}{2} \le \frac{k}{d}. \tag{6}$$

References

- Sjöqvist E, Pati A K, Ekert A, Anandan J S, Ericsson M, Oi D K L and Vedral V (2000) Phys. Rev. Lett. 85 2845–9
- Paweł Horodecki and Artur Ekert (2002) Phys. Rev. Lett. 89, 127902
- New J. Phys. 25 (2023) 043026, Som Kanjilal, Vivek Pandey and Arun Kumar Pati

Measuring Negativity of pure states from single copy

- For a pure bipartite state $|\Psi\rangle = \sum_{j=0}^{d-1}\sqrt{\lambda_j}\,|j\rangle_A|j\rangle_B$, its negativity is given by

$$\mathcal{N}(|\Psi\rangle) = \frac{1}{2}\sum_{i\neq j}\sqrt{\lambda_i\lambda_j}, \tag{7}$$

  where λ_i's are Schmidt coefficients.
- Consider an operator

$$X = \sum_{i,j=0;i\neq j}^{d-1}|i\rangle\langle j| = \sum_{i<j}\underbrace{(|i\rangle\langle j| + |j\rangle\langle i|)}_{G_{ij}}. \tag{8}$$

- For $d=2$ case, X is unitary and the visibility is given by $V = |\,\mathrm{tr}(U\rho_{AB})| = \langle\Psi|\,X_A\otimes X_B\,|\Psi\rangle = 2\mathcal{N}(|\Psi\rangle)$.
- For $d>2$, we show that

$$V = |\,\mathrm{tr}(\rho_{AB}U_{AB})| = \frac{1}{d^2}\left(8\mathcal{N} + (d-2)^2\right), \tag{9}$$

  where $U_{AB} = U_A\otimes U_B$ and $U_K = e^{i\frac{\pi}{d}X_K}$.
- For $d>2$, we show that

$$V = |\,\mathrm{tr}(\rho_{AB}U_{AB})| = \frac{1}{d^2}\left(8\mathcal{N} + (d-2)^2\right), \tag{10}$$

  where $U_{AB} = U_A\otimes U_B$ and $U_K = e^{i\frac{\pi}{d}X_K}$.
- Consider the following mixed state which is a mixture of maximally entangled state and a classical-classical state

$$\rho_{AB} = x\,|\Phi^+\rangle\langle\Phi^+| + \frac{(1-x)}{d}\sum_{j=0}^{d-1}|j\rangle\langle j|\otimes|j\rangle\langle j|. \tag{11}$$

  The entanglement of the above state is given by

$$\mathcal{N}(\rho_{AB}) = \frac{x(d-1)}{2}. \tag{12}$$

- It can be shown that the joint observable $X_A\otimes X_B$, we have $\mathrm{tr}(X_A\otimes X_B\rho_{AB}) = \frac{2\mathcal{N}(\rho_{AB})}{d-1}$. Therefore, the entanglement of the mixed state can be determined from the interferometric setup.

Mutual Predictability from interferometric set-up

- For observables $\mathcal{O}_A$ and $\mathcal{O}_B$ for Hilbert space $\mathcal{H}_A\otimes\mathcal{H}_B$ and the eigenvectors of $\mathcal{O}_A$ and $\mathcal{O}_B$ can be taken as $\{|k\rangle_A\}$ and $\{|l\rangle_B\}$, where $l,k \in \{0,1,2,3,.......d-1\}$. We can define mutual predictability as

$$C_{\mathcal{O}_A,\mathcal{O}_B} = \sum_k {}_B\langle k|_A\langle k|\,\rho_{AB}\,|k\rangle_A|k\rangle_B. \tag{13}$$

- By considering m mutually unbiased observables for each subsystem as $\{\mathcal{O}_{A_i}\}_{i=1}^m$ and $\{\mathcal{O}_{B_i}\}_{i=1}^m$, one obtains

$$\sum_{i=1}^m C_{\mathcal{O}_{A_i},\mathcal{O}_{B_i}} \le 1 + \frac{m-1}{d} \tag{14}$$

  for separable states.
- Mutual predictability for a pair of observables $C_{\mathcal{O}_A,\mathcal{O}_B}$ can be determined an oracle producing the following unitary

$$U_{\mathcal{O}_A,\mathcal{O}_B} = \left(\mathbb{I}\otimes\mathbb{I} - 2\sum_k|k\rangle_A\langle k|\otimes|k\rangle_B\langle k|\right). \tag{15}$$

  We then have

$$C_{\mathcal{O}_A,\mathcal{O}_B} = \frac{1}{2}\left(1 \pm \left|\mathrm{tr}\left(\rho_{AB}U_{\mathcal{O}_A,\mathcal{O}_B}\right)\right|\right). \tag{16}$$

- Measuring visibility for m such unitaries we obtain

$$2\left(1+\frac{(m-1)}{d}\right) - m \ge \sum_{i=1}^m V_i \ge m - 2\left(1+\frac{(m-1)}{d}\right), \tag{17}$$

  for separable states. If $m=2$, then for separable states, we have $V_1 + V_2 \le \frac{1}{d}$.

Witnessing Entanglement of Werner state from Interferometric set-up

- The witness operator $F = \sum_{i,j}|i\rangle\langle j|\otimes|j\rangle\langle i|$ is an entanglement witness operator for Werner state $\rho_w = xQ_s + (1-x)Q_a$, where $Q_s = \frac{2}{d(d+1)}(\mathbb{I}_A\otimes\mathbb{I}_B + F_{AB})$ and $Q_a = \frac{2}{d(d-1)}(\mathbb{I}_A\otimes\mathbb{I}_B - F_{AB})$.
- F_{AB} is also an unitary operator, we also have

$$\mathrm{tr}(U\rho_w) = \begin{cases} 2(2x-1)e^{i\pi} & \text{when } \rho_w \text{ is entangled} \\ 2(2x-1) & \text{when } \rho_w \text{ is separable.}\end{cases} \tag{18}$$

  For entangled Werner state ($x < \frac{1}{2}$), the phase shifts from ϕ to $\phi + \pi$.

Conclusion

- Here, we demonstrate how to measure the linear entropy, negativity and the Schmidt number of bipartite systems from the visibility of Mach–Zehnder interferometer using single copies of the quantum state.
- We also propose how to measure the mutual predictability experimentally from the intensity patterns of the interferometric set-up without having to resort to local measurements of mutually unbiased bases.
- Furthermore, we show that the entanglement witness operator can be measured in a interference setup and the phase shift is sensitive to the separable or entangled nature of the state.