Topology for qLDPC: transversal non-Clifford gates and magic state fountain on homological product codes with constant rate and beyond the $N^{1/3}$ distance barrier

Guanyu Zhu

I. Introduction: Rapid progress in quantum low-density parity-check (qLDPC) codes has been made in recent years to achieve asymptotically optimal quantum information storage [1]. On the other hand, fault-tolerant quantum computation requires also efficient schemes to implement parallelizable logical gates, with the most challenging part being the non-Clifford gates. For more than a decade ever since Bombin's 3D color codes [2] (see also [3, 4]), there existed a long-standing $N^{1/3}$ distance barrier for transversal non-Clifford gates in topological stabilizer codes and more generally qLDPC codes. For topological stabilizer codes defined on Euclidean geometries, this is constrained by the Bravyi-König (BK) bound [5].

In this talk, we present a new strategy to break this distance barrier from Ref. [6]. In particular, we develop a topological theory for qLDPC codes, showing the hidden simplicial or CW complex structure for arbitrary qLDPC and CSS codes based on product constructions, which hence admit cohomology operations such as cup products. One hence uses them to implement transversal CCZ gates on a 3D homological product code on a simplicial or CW complex with constant rate and $\Omega(\sqrt{N})$ distance (further extended to $\Omega(N^{2/3})$ in Ref. [7]), where the distance barrier is broken through the non-Euclidean geometries from the fibre bundle [8, 9, 10]. This can be further used for fault-tolerantly preparing $\Omega(\sqrt{N})$ independent logical magic states without distillation ('*magic state fountain*' [11, 6]).

II. Cup product and cohomology operation on simplicial complexes: For a CSS code defined on a simplicial complex $\mathcal{L}$ with qubits placed on a q-simplex, one can introduce operator-valued $\mathbb{Z}_2$ q-cochain $\hat{a}^q$, with its value on each q-simplex s_q corresponding to the Pauli-Z operator: $(-1)^{\hat{a}^q(s_q)} = Z(s_q)$. Its eigenstate is the q-cochain state in the diagonal basis, i.e., $\hat{a}^q |c^q\rangle = c^q |c^q\rangle$, with $c^q \in C^q(\mathcal{L};\mathbb{Z}_2)$. We then introduce the cup product as the bilinear map on the cochain groups $\cup : C^p \times C^q \to \mathcal{C}^{p+q}$. Now we can define a cohomology operation on the operator-valued cochains from three identical copies of codes as $U = (-1)^{\int_\mathcal{L} \hat{a}^{q_1}_{(1)} \cup \hat{a}^{q_2}_{(2)} \cup \hat{a}^{q_3}_{(3)}}$, where $\int_\mathcal{L}$ represents the discrete sum over $\mathcal{L}$ [11, 12]. It corresponds to a nearly transversal gate (constant-depth circuit) composed of a product of local CCZ gates acting on the three copies. The coboundary invariance property of U guarantees that it preserve the code space $\mathcal{C}$. When expanding $\hat{a}^{q_i}_{(i)}$ in the cocycle basis $\{\alpha^{q_1}\}$, $\{\beta^{q_2}\}$, and $\{\gamma^{q_3}\}$, we obtain $U = \prod_{\alpha,\beta,\gamma}(-1)^{\int_\mathcal{L}(\hat{n}_\alpha \alpha^{q_1}) \cup (\hat{m}_\beta \beta^{q_2}) \cup (\hat{l}_\gamma \gamma^{q_3})} = \prod_{\alpha,\beta,\gamma} \overline{\text{CCZ}}[\alpha,\beta,\gamma]^{\int_\mathcal{L} \alpha^{q_1} \cup \beta^{q_2} \cup \gamma^{q_3}}$. The unitary U hence applies a logical CCZ gate on any triplet of logical qubits whose logical-X operator support (cocycles) have a triple intersection $\int_\mathcal{L} \alpha^{q_1} \cup \beta^{q_2} \cup \gamma^{q_3} = 1$.

III. Construction via the code-to-manifold mapping: Now in order to achieve constant rate and polynomial distance, we consider taking homological product of asymptotically good classical or quantum codes [13]. However, these codes are typically defined on general chain complexes, which do not necessarily admit cup products. We can consider introducing the simplicial complex structure to these codes. One potential obstacle is that if one defines classical codes on a graph (1D simplicial complex) with bits put on the edges (without local codes), it cannot be asymptotically good since a constant-rate code will only have a logarithmic upper bound in distance [14]. The way out is to go to higher dimensional topological expanders defined on higher dimensional simplicial complex. Indeed classical codes defined on random higher-dimensional simplicial complexes where bits are placed on higher simplices with dimension higher than 2 is proven to be asymptotically good [15], altough not LDPC. Instead, we build upon the recent breakthrough by Freedman and Hastings (FH) on geometrizing a quantum code into an 11-manifold [16]. We generalize such an approach by also mapping an arbitrary classical LDPC code to a manifold, and then use the manifolds built from classical or quantum codes as legos to build product quantum codes. The main difference from the other independent work along this direction [17, 18, 19] is that our construction is fully topological and does not need any local combinatorial condition such as the multiplication property. Also, our code is geniunely LDPC in contrast to *polylog*(N) stabilizer weight in [17].

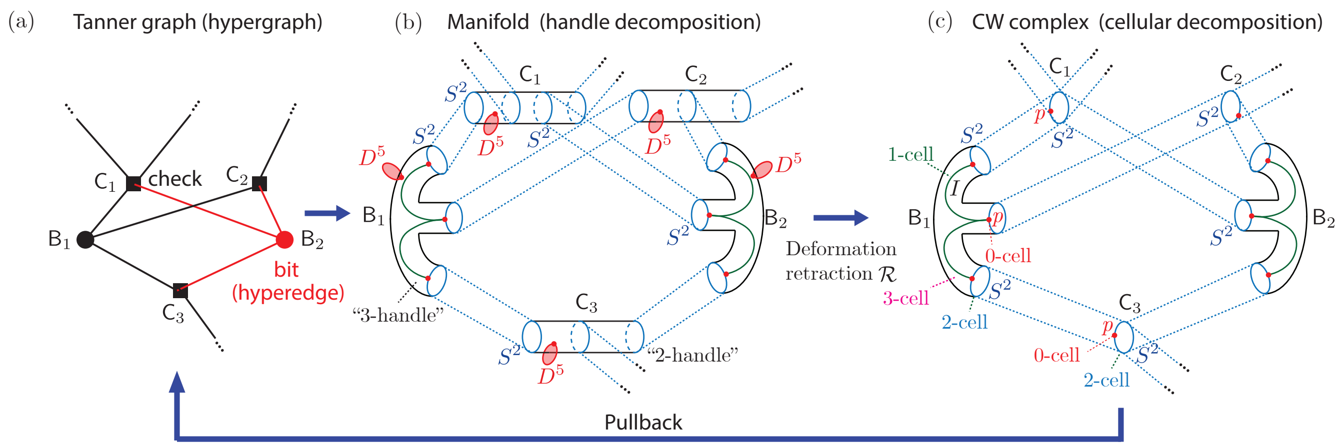

Figure 1

The construction starts with lifting the $\mathbb{Z}_2$ chain complex associated with the skeleton classical code $\bar{\mathcal{C}} = \text{Ker}(\bar{\mathsf{H}}) = \text{Ker}(\partial)$ to a chain complex with $\mathbb{Z}$ coefficients: $\hat{C}_1 \xrightarrow{\hat{\partial}=\hat{\bar{\mathsf{H}}}} \hat{C}_0$, where $\hat{\partial} = \hat{\bar{\mathsf{H}}}$ is the lifted boundary map and parity check matrix. Here, a naive lift (0 mod 2 $\rightarrow$ 0, 1 mod 2 $\rightarrow$ 1) suffices. We then use the handle construction to "thicken" the Tanner graph of the classical code to a manifold by mapping each bit and check to a dressed k-handle and dressed $(k-1)$-handle respectively, which are themselves composed of handles with the same and lower indexes (denoted by "k-handle" from now on for simplicity). In r dimension each "k-handle" has the form $\tilde{h}_k = N_k \times D^{r-k}$, where D^m represents the m-dimensional disk (ball) and N_k is the dressed core. In order to separate with the short spurious 1-(co)cycle, we first choose $k = 3$ and $r = 8$, with the "2-handle" (check) being $\mathsf{C}_j = S^2 \times D^6$ (S^2 stands for 2-sphere) and the "3-handle" (bit) being a thickened punctured 3-sphere $\mathsf{B}_i = (S^3 \backslash \sqcup_{m=1}^{f(i)} D^3_m) \times D^5$, where $f(i)$ is the number of checks that the bit B_i couples to, as illustrated in Fig. 1(a,b). We then attach the "k-handles" to the boundary of "$(k-1)$"-handles with the attaching map specified by the lifted boundary map $\hat{\partial}$, which gives a 3-handlebody H (a manifold with boundaries). We hence obtain the right portion of the following handle chain complex $\mathcal{L}_h$:

$$C_8 \rightarrow C_7 \rightarrow C_6 \xrightarrow{\hat{\partial}^T \sim \bar{\mathsf{H}}^T} C_5 \rightarrow 0 \rightarrow \underset{\text{check}}{C_3} \xrightarrow{\hat{\partial} \sim \bar{\mathsf{H}}} \underset{\text{bit}}{C_2} \rightarrow C_1 \rightarrow C_0, \tag{1}$$

where the middle 4-chain group C_4 is trivial. To get the left portion, we take another identical upside-down copy H^*, where the k-handles becomes their dual handles, i.e., the $(8-k)$-handles. We then take the *double* of the 3-handlebody by gluing the two identical copies along their common boundary: $\mathcal{D}H = H \cup_{\partial H} H^*$, which kills the boundary and results in a closed orientable 8-manifold $\mathcal{M}^8$. The *doubling* introduces Poincaré duality, which is absent in the input code $\bar{\mathcal{C}}$, as reflected from the isomorphism between the left and right portion with dual dimensions. We have proved the following theorem:

Theorem 2: *From the Tanner graph of any input skeleton classical LDPC code $\bar{\mathcal{C}}$ = Ker($\bar{\mathsf{H}}$) that is asymptotically good, one can define a classical LDPC code $\mathcal{C} = H_3(\mathcal{L}^r; \mathbb{Z}_2)$ on the triangulation $\mathcal{L}^r$ of the manifold $\mathcal{M}^r$ (with bits and checks placed on the 3- and 2-simplices) that is asymptotically good (for $r \geq 8$).*

This is interesting in its own right, since it is the first asymptotically good classical LDPC code defined on a high-dimensional simplicial complex $\mathcal{L}^r$ without local codes.

We then show that using the subsystem code idea that treat the spurious (co)cycles as gauge qubits (Lemma 6), we can further lower the dimension of the manifold to $r = 4$ with bits and checks placed on the 2- and 1-simplices respectively. This gives rise to a construction of a thickened 3D hypergraph product code defined on the 12-manifold $\widetilde{\mathcal{M}}^{12}$ as the product of three 4-manifolds built from good classical codes with $O(1)$ stabilizer weight, constant rate and $\Omega(N^{1/3})$ distance. The code admits a cohomology operation between three copies of codes $U = (-1)^{\int_{\widetilde{\mathcal{M}}^{12}} \hat{a}^4_{(1)} \cup \hat{a}^4_{(2)} \cup \hat{a}^4_{(3)}}$ which implements $\Theta(N)$ logical CCZ given by the $\Theta(N)$ triple intersections (Theorem 4) originated from Poincaré duality. The same property holds for the higher-dimensional construction of the product of three 8-manifolds mentioned above (Theorem 5). Now to break the $N^{1/3}$ distance barrier, we take a product of thickened good classical and quantum LDPC codes, with the later defined on the 11-manifold from the FH mapping [16] taking the input skeleton code $\mathcal{C}$ as the balanced (lifted) product code constructed by Panteleev and Kalachev [1], which is geometrically a highly non-Euclidean fibre bundle [8, 9]. We hence reach the main theorem:

Theorem 7: *There exist a family of thickened 3D homological product codes $\tilde{\mathcal{C}}$ defined on the triangulation of a 15-manifold $\widetilde{\mathcal{M}}^{15}$ with encoding rate $K = \Theta(N)$, subsystem-code distance $D = \Omega(\sqrt{N})$ and constant stabilizer weight $w = O(1)$, such that a constant-depth circuit implementing the cohomology operation of a triple cup product can give rise to $\Theta(N)$ logical CCZ gates on $\tilde{\mathcal{C}}$.*

When initializing all the logical qubits in the $\overline{|+\rangle}$ states and apply logical CCZ's, we obtain a hypergraph magic state [11, 6] as computation resource with its structure determined by the triple-intersection property, which can also contain a large amount of magic and can be hard for classical simulation [20]. Moreover, when setting a portion of the logical qubits to the $\overline{|0\rangle}$ state, one can fault-tolerantly prepare $\Theta\sqrt{(N)}$ independent logical CCZ magic states with $\Theta(\sqrt{N})$ distance without the need for distillation ('magic state fountain'), a concept introduced in Ref. [11, 6]. This significantly outperform a $[[N, 1, \Omega(N^{1/3})]]$ 3D color code [2] which can only prepare 1 magic state with distance $\Omega(N^{1/3})$. We further extend the scheme to a product of three good qLDPC codes in Ref. [7], obtain $\Omega(N^{2/3})$ distance, with dimension $\Theta(N^{2/3})$ which allows one to prepare $\Theta(N^{1/3})$ independent CCZ magic states.

IV. Hidden CW complex structure, pullback to the input skeleton complex, and practicality for near-term implementation: The above discussion focuses on the asymptotic regime, while cautious readers may question the practicality in near-term implementation since the dimension looks high and the overhead of subdividing the manifold into triangulations may be a large constant. To eliminate this worry, we further show that one can *deformation retract* the manifold $\mathcal{M}^r$ to a cellular chain complex $\mathcal{L}_c$ (often called CW complex [21]) as hinted in Ref. [16] and elaborated in Ref. [22], which is isomorphic to the handle chain complex $\mathcal{L}_h$ in Eq. (1). Each k-handle $h_k = D^k \times D^{r-k}$ is retracted to its core—the k-cell D^k, and the dressed "k-handle" $\tilde{h}_k$ is retracted to the dressed core N_k [Fig. 1(c)]. Each bit and check corresponds to a unique 3-cell and 2-cell. Due to the isomorphism, the classical or quantum code $\mathcal{C}$ defined on the CW complex is completely the same as the skeleton code $\bar{\mathcal{C}}$, while the hidden CW complex structure has mathematically well-defined cup product which can implement logical gates. Note the code $\mathcal{C}$ on the CW complex is non-topological, since $\mathcal{L}_c$ is no longer a discretization of the manifold like the triangulation. This compact realization makes the near-term implementation practical. Interestingly, the isomorphism between $\mathcal{L}_c$ and $\mathcal{L}_h$ presdrves the Poincaré duality and $\mathcal{L}_c$ is hence a *Poincaré complex.*

In the follow-up work in Ref. [23], we have further shown how to perform a pullback of the higher-dimensional CW complex to the 2D skeleton input chain complex, along with the cup product structure and hence transversal CCZ gates directly acting on the skeleton complex, which hence achieves minimal overhead. In this simplification, it is shown that the cup product can be translated to the chain-cochain pairing of the skeleton chain complex, with the Poincaré duality corresponding to the cyclce-cocycle duality.

Finally, we note that this scheme can be generalized to balance product codes [24] to further minimize the overhead by moding out the underlying symmetries. The scheme is also applicable to arbitrary skeleton classical or quantum code without requiring any specific structure. It should hence be achievable with near-term codes with O(50) to O(1000) qubits, such as the homological product of a bivariate bicycle code [25] and a classical expander code, or the tricycle codes based on balanced product construction [26, 27]. The advantage of the present scheme is that it is widely applicable (flexible) and scalable, which does not require intensive numerical search at all. In particular, the logical gate structure can be determined analytically for arbitrary system size with controllable scaling, i.e., $\Theta(\sqrt{k}) = \Theta(\sqrt{n})$ disjoint logical CCZ magic states, while other follow-up schemes based on the numerical search are currently limited to both small distance and only $O(1)$ disjoint logical CCZ, or large depth of the physical CCZ circuit [26, 27].

References

[1] P. Panteleev and G. Kalachev, "Asymptotically good quantum and locally testable classical ldpc codes," in *Proc. ACM STOC*, (New York, NY, USA), pp. 375—388, Association for Computing Machinery, 2022.

[2] H. Bombin, "Gauge color codes: optimal transversal gates and gauge fixing in topological stabilizer codes," *New Journal of Physics*, vol. 17, pp. 083002–14, Aug. 2015.

[3] A. Kubica, B. Yoshida, and F. Pastawski, "Unfolding the color code," *New Journal of Physics*, vol. 17, p. 083026, Aug. 2015.

[4] M. Vasmer and D. E. Browne, "Three-dimensional surface codes: Transversal gates and fault-tolerant architectures," *Phys. Rev. A*, vol. 100, p. 012312, Jul 2019.

[5] S. Bravyi and R. König, "Classification of Topologically Protected Gates for Local Stabilizer Codes," *Phys. Rev. Lett.*, vol. 110, pp. 170503–5, Apr. 2013.

[6] G. Zhu, "A topological theory for qldpc: non-clifford gates and magic state fountain on homological product codes with constant rate and beyond the $n^{1/3}$ distance barrier," *arXiv preprint arXiv:2501.19375*, 2025.

[7] G. Zhu, "Transversal non-clifford gates on qldpc codes breaking the $\sqrt{N}$ distance barrier and quantum-inspired geometry with systolic freedom," *arXiv preprint arXiv:2507.15056*, 2025.

[8] M. H. Freedman, D. A. Meyer, and F. Luo, "Z2-systolic freedom and quantum codes," in *Mathematics of quantum computation*, pp. 303–338, Chapman and Hall/CRC, 2002.

[9] M. Hastings, J. Haah, and R. O'Donnell, "Fiber bundle codes: breaking the $n^{1/2}\mathrm{polylog}(n)$ barrier for quantum ldpc codes," in *Proc. ACM STOC*, (New York, NY, USA), pp. 1276–1288, Association for Computing Machinery, 2021.

[10] N. P. Breuckmann and J. N. Eberhardt, "Balanced Product Quantum Codes," *arXiv:2012.09271*, 2020.

[11] G. Zhu, S. Sikander, E. Portnoy, A. W. Cross, and B. J. Brown, "Non-clifford and parallelizable fault-tolerant logical gates on constant and almost-constant rate homological quantum ldpc codes via higher symmetries," *arXiv:2310.16982*, 2023.

[12] M. Barkeshli, Y.-A. Chen, S.-J. Huang, R. Kobayashi, N. Tantivasadakarn, and G. Zhu, "Codimension-2 defects and higher symmetries in (3+1)D topological phases," *SciPost Phys.*, vol. 14, p. 065, 2023.

[13] S. Bravyi and M. B. Hastings, *Homological product codes.* the 46th Annual ACM Symposium, 2014.

[14] R. Meshulam, "Graph codes and local systems," *arXiv*, 2018.

[15] J. McCullough and H. Newman, "Asymptotically good homological error correcting codes," *Journal of Algebra Combinatorics Discrete Structures and Applications*, vol. 6, no. 3, pp. 135–145, 2019.

[16] M. Freedman and M. B. Hastings, "Building manifolds from quantum codes," *arXiv:2012.02249*, 2020.

[17] L. Golowich and T.-C. Lin, "Quantum ldpc codes with transversal non-clifford gates via products of algebraic codes," *arXiv preprint arXiv:2410.14662*, 2024.

[18] T.-C. Lin, "Transversal non-clifford gates for quantum ldpc codes on sheaves," *arXiv preprint arXiv:2410.14631*, 2024.

[19] N. P. Breuckmann, M. Davydova, J. N. Eberhardt, and N. Tantivasadakarn, “Cups and gates i: Cohomology invariants and logical quantum operations,” *arXiv preprint arXiv:2410.16250*, 2024.

[20] J. Chen, Y. Yan, and Y. Zhou, “Magic of quantum hypergraph states,” *arXiv preprint arXiv:2308.01886*, 2023.

[21] A. Hatcher, *Algebraic Topology.* Cambridge; New York: CUP, 2001.

[22] V. Guemard, “Lifting a css code via its handlebody realization,” *arXiv preprint arXiv:2505.14327*, 2025.

[23] G. Zhu, R. Kobayashi, and P.-S. Hsin, “Non-abelian qldpc: Tqft formalism, addressable gauging measurement and application to magic state fountain on 2d product codes,” *arXiv preprint arXiv:2601.06736*, 2026.

[24] N. P. Breuckmann and S. Burton, “Fold-transversal clifford gates for quantum codes,” *arXiv preprint arXiv:2202.06647*, 2022.

[25] S. Bravyi, A. W. Cross, J. M. Gambetta, D. Maslov, P. Rall, and T. J. Yoder, “High-threshold and low-overhead fault-tolerant quantum memory,” *Nature*, vol. 627, no. 8005, pp. 778–782, 2024.

[26] A. Jacob, C. McLauchlan, and D. E. Browne, “Single-shot decoding and fault-tolerant gates with trivariate tricycle codes,” *arXiv preprint arXiv:2508.08191*, 2025.

[27] V. Menon, J. P. Bonilla-Ataides, R. Mehta, D. B. Tan, and M. D. Lukin, “Magic tricycles: efficient magic state generation with finite block-length quantum ldpc codes,” *arXiv preprint arXiv:2508.10714*, 2025.