# Energy, Bosons and Computational Complexity

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Traditionally, much of theoretical quantum algorithms development, from Shor's integer factoring algorithm [Sho94] in 1994 to the Quantum Singular Value Transform framework (QSVT) [GSLW19] of 2019, has taken place in the *Discrete-Variable (DV)* computational model, i.e. based on a system of finite-dimensional qu$d$its for some $d \in O(1)$. While its "closeness" to traditional classical gate-based computing and abstraction of the underlying implementation gained researcher's preference, the advent of the Noisy Intermediate Scale Quantum computing era has shifted focus onto current experimental limitations on qubit counts, coherence, and circuit depth and have made it clear that the computational models need to become "more aligned" with the underlying physics of experimental platforms. To do so, attention in the algorithms community is shifting back to the origins of quantum information theory: *Continuous-Variable (CV)* systems, i.e. systems described as having continuous degrees of freedom, and thus infinite-dimensional, such as bosonic systems.

**Why study the power of quantum computation on CV systems?** First, many quantum systems are inherently CV in nature. For example, the two recent experimental platforms for quantum advantage implementations of Gaussian Boson Sampling [AA11; HKS+17; ZPL+19] and Random Circuit Sampling [AAB+19; BFNV19] are based on quantum photonics (which is CV) and superconducting qubits (which use CV systems to simulate DV systems). Second, experimental platforms such as photonics have a decades-long history of successful implementations, like in Quantum Key Distribution [BG84]. Third, *hybrid* DV-CV platforms (e.g. superconducting, trapped ion, and neutral atom [LSS+24]) are rising as potential experimental platforms, which has led to intriguing theoretical algorithmic developments, such as the CV quantum factoring algorithm of Brenner, Caha, Coiteux-Roy and Koenig (BCCK) [BCCK24] which requires only three oscillators and a qubit, i.e. a *constant* number of quantum systems.

**Energy.** The fact that the BCCK [BCCK24] CV factoring algorithm requires only $O(1)$ space raises the question — *where did the exponential degrees of freedom in an $n$-qubit DV quantum system go?* Well, into the *energy* $E$, i.e. into exponentially many degrees of freedom in the photon number/Fock basis. Indeed, it has long been known that the infinite-dimensional nature of CV systems allows one to do seemingly impossible things, such as the dense coding channel capacity tending to infinity as $E \to \infty$ [BK00]. Experimentally, exponential energy in the lab would appear infeasible, let alone *infinite* energy. But can one formally *prove where the boundary between "plausibly feasible" and "intractable" energy levels* lies? And what might be the *implications for experimental CV setups*?

**Studying energy via computational complexity.** We address these questions with computational complexity theory, characterizing which resources (e.g. time, space, communication, and crucially for us *energy*) are necessary and/or sufficient to solve tasks in bosonic computation. At a high level, our computational model, dubbed the CV model, generalizes that of [LB99; CJMM25]. It compares to a standard DV setup/"DV model" as follows: the qubit state space is replaced by the infinite-dimensional space $(\mathbb{C}^\infty)^{\otimes n} \cong \ell^2(\mathbb{N}_0^n, \mathbb{C})$, the set of square-summable complex sequences; the input state is the vacuum state denoted $|0\rangle$; gates are $k$-mode unitary $e^{iHt}$ for Hamiltonian $H$, evolution time $t$, and locality $k \in O(1)$, where $H$ is a $O(1)$-degree polynomial in the position $\hat{X}$ and momentum $\hat{P}$ operators on $k$ modes with constant coefficients (e.g. Gaussian Hamiltonians which are degree-2 polynomials) and the total runtime of a sequence of gates $\Pi_{j=1}^{m} e^{it_j H_j}$ is $\sum_{j=1}^{m} t_j$; and finally, single-mode photon-number measurements. In this model, there exists a gate set (Gaussian gates and the cubic phase gate $H = X^3$) so that the CV model with polynomial runtime is contained in $\mathsf{EXPSPACE}$ [CJMM25]. The latter has since been improved to $\mathsf{PSPACE}$ [URC25].

**Interpreting these facts.** That $\mathsf{BQP} \subseteq \mathsf{AWPP}$ [FR99], whereas the best upper bound known on poly-time CV models with certain *specific* gate sets is $\mathsf{PSPACE}$ (which is believed to strictly contain $\mathsf{BQP}$ [Vya03]), speaks to the difficulty of analyzing infinite-dimensional systems. Indeed, in contrast to the DV setting, a CV analogue of the Solovay–Kitaev theorem [DN05] is only known to hold for specific subclasses of gate sets [BDLR21; ABC25]. This raises many questions: *What role does energy play in all of this? Can energy be used to disprove versions of a CV Solovay–Kitaev theorem, e.g. by differentiating the power of different gate sets? Can one obtain formal evidence as to whether constant mode, exponential energy setups such as BCCK [BCCK24] are "reasonable" or not? And is it possible that even poly-time CV computations are more powerful than BQP?*

**RESULTS.** Our work addresses this in the context of bosonic quantum computations by covering three interconnected themes: (1) Why energy is a problem to begin with (i.e. quantifying the rate of energy growth in bosonic computations), (2) what energy buys us computationally (i.e. complexity-theoretic lower bounds), and (3) when we can bypass the energy barrier (i.e. simulation algorithms and complexity-theoretic upper bounds). Combining results from (2) and (3) yields a no-go for certain versions of a CV Solovay-Kitaev theorem, which is discussed last. We provide informal statements of our main contributions here and refer to the Technical Manuscript for details.

**1. Quantifying the rate of energy growth.** We begin by formally proving that energy in CV systems (i.e. $\langle\psi| \hat{N} |\psi\rangle$) can grow at an incredible rate with the right gate sets, even when restricted to polynomial or *constant*(!) time. We show:

**Theorem** (Energy growth rates). *For CV circuits of size $t$: (1) Any circuit with Gaussian and Kerr gates $H = \hat{N}^2$ has energy at most $e^{O(t)}$. (2) Energy in single-mode circuits from Gaussian and cubic phase gates can grow as fast as $\sim (2^t)^{2^t}$. (3) The state $|\psi\rangle = e^{-itH} |0\rangle_0 |0\rangle_1$ with $H = \frac{i}{2}\hat{X}_0^4(\hat{a}_1^2 - \hat{a}_1^{\dagger 2})$ has infinite energy for any $t > 0$.*

These are to be interpreted as follows: (1) demonstrates that, in contrast to what is to follow, not all non-Gaussian gate sets lead to drastic energy growth; Gaussian gates and the Kerr give at most exponential energy in poly-time. (2) then rigorously proves that even well-studied gate sets such as Gaussian gates + cubic phase gate [GKP01; SV11] already yield provably doubly exponential energy in polynomial time. (3) demonstrates the most extreme case; even *constant* time can suffice to achieve infinity energy, i.e. $\langle\psi| \hat{N} |\psi\rangle$ diverges. In sum, this result shows that energy growth rates, even in small CV circuits, indeed need be reckoned with carefully.

**2. Complexity-theoretic lower bounds.** Having formally demonstrated rapid energy growth in the CV model, we next prove complexity-theoretic implications thereof. Define $\mathsf{CVBQP}$ as the set of promise problems solvable in the CV model with (1) *polynomial* runtime and (2) whereas in principle no energy bounds are assumed *during* the computation, at the time of measurement we are promised the energy is at most *polynomial*. Since universal gate sets are not known for the CV model, we further use $\mathsf{CVBQP}[\mathcal{G}]$ to denote $\mathsf{CVBQP}$ with finite gate set $\mathcal{G}$.

**Theorem** (The computational power of energy). *There exists a finite gate set $\mathcal{G}$ for each of the following such that: (1) $\mathsf{NP} \subseteq \mathsf{CVBQP}[\mathcal{G}]$ with at most exponential energy and $O(1)$ modes. (2) For all $k \in \mathbb{N}$, $\mathsf{NTIME}(\exp^{(k)}(n)) \subseteq \mathsf{CVBQP}[\mathcal{G}]$ with $O(k)$ modes. (3) $\mathsf{PTOWER} \subseteq \mathsf{CVBQP}[\mathcal{G}]$.*

(1) shows $\mathsf{CVBQP}$ with at most exponential energy and $O(1)$ modes already suffice to capture $\mathsf{NP}$. This yields formal evidence that the BCCK CV factoring algorithm's [BCCK24] computational model is perhaps *not* "reasonable", as recall BCCK *also* uses exponential energy and $O(1)$ modes. We caution that these results are gate set dependent, and our gate sets differ from BCCK. (2) shows that with only $k$ modes but potentially unbounded energy, $\mathsf{CVBQP}$'s power blows up to non-deterministic $\exp^{(k)}(n)$ time, for $k$-fold iterated exponential $\exp^{(k)}(n) := \exp(\exp(\cdots \exp(n) \cdots))$. (2) now yields (3), which shows that with no energy or sub-polynomial mode restrictions on $\mathsf{CVBQP}$, we can even solve $\mathsf{PTOWER}$, the set of languages decidable in deterministic time $\exp^{(\mathrm{poly})}(n)$, i.e. a power tower of polynomial height. In sum, the "wrong" choice of gate set can yield rapid energy growth, which in turn provides immense computational power. We further prove an oracle-separation-based *energy hierarchy theorem*, similar to time hierarchy theorems such as $\mathsf{DTIME}(n^k) \subsetneq \mathsf{DTIME}(n^{k+1})$ for $k \geq 1$, showing that even with mild assumptions on the gate set, more energy provably yields greater computational power in the oracle setting (see Theorem 1.3).

Furthermore, high energy makes computing *properties* of CV computations difficult — undecidable, in fact! For instance, we prove that it is undecidable whether a Gaussian state evolved under a polynomial Hamiltonian has finite energy (see Theorem 1.4). So not only can the right gate set rapidly yield infinite energy, but even deciding if it has this property is undecidable. Notwithstanding, we show that any uniform finite (but unknown) energy bound on $\mathsf{CVBQP}$ with "physical computations" suffices for decidability (see Theorem 1.6).

**3. Simulation algorithms and upper bounds.** Finally, we prove upper bounds and find conditions under which the CV model can be simulated within standard complexity classes. We begin with the natural question — is $\mathsf{CVBQP}$ with polynomial energy simulatable by $\mathsf{BQP}$?

**Theorem.** $\mathsf{CVBQP}$ *with polynomial energy and any gate set is in* $\mathsf{BQP}/_{\mathrm{poly}}$.

If an additional assumption holds for the gate set used, (which holds for the Kerr, Gaussians, Jaynes–Cumming, and cubic phase gate), then the bound improves to $\mathsf{BQP}$. More generally, *any* CV model circuit on $n$ modes, $k$-local gates, evolution time $T$ and energy bound $E$ can be simulated with some DV quantum circuit of width $O(n \log \frac{E^*T}{\delta})$ and depth $O(TE^{*2k})$. We also refine these results for the case of the Gaussian + cubic gate set, showing that even without an energy bound, the model is contained in $\mathsf{PP}$ (see Thm 1.7)—improving the previous $\mathsf{PSPACE}$ upper bound [URC25] in a key manner, in that the computational power of this gate set is provably "not too far" from $\mathsf{BQP}$ ($\mathsf{BQP} \subseteq \mathsf{AWPP} \subseteq \mathsf{PP}$). Equally interesting is the novel proof technique for this, which develops a finite energy CV state injection gadget for the cubic gate.

**4. No-go for versions of CV Solovay–Kitaev.** Combining our complexity class lower and upper bounds, we obtain that, unless $\mathsf{PTOWER} \subseteq \mathsf{PP}$, the Gaussian+cubic phase gate set cannot be an *efficient* universal gate set in the manner of a CV Solovay–Kitaev. Our no-go result motivates the future study of explicit rigorous compilation results to experimentally realizable CV gates.

**OUTLOOK.** Our work highlights energy as a defining resource in bosonic computation, with its growth fundamentally shaping the computational landscape. We identify three key takeaway messages: (1) in CV systems, energy can grow at rates far more dramatic than time or space in DV models, (2) high-energy bosonic computations offer surprising computational power, but in physically infeasible ways, and (3) under reasonable energy constraints, bosonic quantum computations can be simulated within $\mathsf{BQP}$. Our findings show that, akin to time and space in classical and DV quantum computation, energy is a fundamental resource in bosonic quantum computational complexity.

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