The Learnability of BosonSampling States

J. T. Iosue, Y. Wang, I. Datta, S. Ghosh, C. Oh, B. Fefferman, A. V. Gorshkov

The intrinsic exponential sample and time complexity of quantum state tomography as system size scales [1–5] motivates the search for natural classes of quantum states that can be learned efficiently with a small number of samples and modest computation. Existing learning algorithms often rely crucially on classical simulability (e.g., [6, 7]). Do there exist efficient learners even when classical simulation is hard, e.g. for quantum advantage experiments? While prior results address this question for some states generated by IQP circuits [8] and random quantum circuits at sufficiently low depth [9–11], this question has remained open for Fock states acted upon by Gaussian unitaries, as in BosonSampling [12]. In standard BosonSampling, an n mode Fock state is acted upon by a linear optical unitary. Throughout this work, we consider more general states—namely Fock states acted upon by an arbitrary Gaussian unitary. We will refer to these as *BosonSampling states*. We ask:

Does there exist a sample- and time-efficient algorithm to learn any BosonSampling state?

In this work, we answer this question in the affirmative, thereby resolving an open problem posed by Aaronson and Grewal [13]. We prove this in two parts. First, we show informational completeness, proving that BosonSampling states are fully characterized by their first four moments, by introducing a formalism of symplectic invariants—functions of the moments of a bosonic state unchanged by Gaussian operations (i.e., symplectic transformations). We expect that this formalism, developed in much greater generality in the manuscript, may be of independent interest. As a special case we obtain:

Theorem 1 (Informational completeness)**.** *Every BosonSampling state is completely characterized by its first four moments.*

Theorem 1 alone does not suffice to resolve our motivating question, as quantum learning theory is ripe with examples of information-computation gaps in which tomographically-complete information cannot be extracted efficiently (see e.g. [14, 15]). Indeed, there are few algorithms that are able to exploit higher-order moments for faster learning. Our second result shows that we can nonetheless close this gap and give an efficient algorithm to learn BosonSampling states generated by any arbitrary Gaussian unitary, including linear optical unitaries as a special case. For a state of n bosons, our algorithm uses $\mathrm{poly}(n)$ copies of the state, $\mathrm{poly}(n)$ time, and learns the state to inverse polynomial fidelity.

Theorem 2 (Algorithmic efficiency, Informal)**.** *Let $|\psi\rangle$ be an arbitrary n mode BosonSampling state; that is, $|\psi\rangle = \mathcal{U}|\boldsymbol{f}\rangle$ for an arbitrary unknown Gaussian unitary $\mathcal{U}$ and an arbitrary unknown Fock state $|\boldsymbol{f}\rangle$. Choose a desired error tolerance ε. Then our algorithm finds a Gaussian unitary $\mathcal{V}$ and a $\boldsymbol{g}$ such that*

$$|\langle \boldsymbol{f}|\mathcal{U}^\dagger\mathcal{V}|\boldsymbol{g}\rangle| \geq 1 - \mathrm{poly}(n, f_{\max}, \varepsilon)\,, \tag{1}$$

where $f_{\max} = \max_i f_i$. The algorithm has a sample- and time-complexity of $\mathrm{poly}(n, f_{\max}, 1/\varepsilon)$.

The technical manuscript provides explicit error bounds in place of "$\mathrm{poly}(n, f_{\max}, \varepsilon)$" and "$\mathrm{poly}(n, f_{\max}, 1/\varepsilon)$".

Technical obstacles and comparison to related work. Aaronson and Grewal pose the learnability of BosonSampling states as an open problem in their work resolving a similar question for free (i.e., Gaussian) fermions [13]. They note that generalizing their approach to BosonSampling states, which

	Sample complexity	**Time complexity**
Learning t-doped Gaussian states [16]	$\exp(n)$	$\exp(n)$
Learning pure "Gaussian-entanglable" states via classical shadows [14]	$\mathrm{poly}(n)$	$\exp(n)$
(This work) Learning BosonSampling states	$\mathrm{poly}(n)$	$\mathrm{poly}(n)$

Table 1: Summary of algorithms applied to BosonSampling states. We note that our algorithm specifically applies to BosonSampling states, while the algorithms in [14, 16] apply to other settings.

are non-Gaussian, yields a complicated set of nonlinear equations. Theorem 2 thus resolves this open problem.

We provide a comparison of other learning algorithms applied to BosonSampling states in Table 1. Recent work [16–18] gives efficient algorithms for learning Gaussian states—which in particular shows the learnability of Gaussian BosonSampling states. By contrast, our algorithm learns a class of non-Gaussian states. Specifically, [16] shows that learning general non-Gaussian states is inefficient in sample and time complexity, in addition to unfavorable trace-distance error scaling. They then introduce the formalism of t-doped Gaussian states, those generated by the action on $|0\rangle^{\otimes n}$ of unitary composed of interleaving $t+1$ Gaussian unitaries with t non-Gaussian unitaries. Finally, they give an algorithm to learn t-doped Gaussian states using $\sim n^t$ copies. Importantly, Fock states are non-Gaussian and can be thought of as n-doped Gaussian states—thus the algorithm in [16] learns BosonSampling states using $\exp(n)$ copies. By contrast, our algorithm uses only $\mathrm{poly}(n)$ copies and $\mathrm{poly}(n)$ time, providing an exponential reduction in sample and time complexity.

Additionally, [18] gives a learning algorithm for Gaussian states whose sample complexity is independent of energy or particle number. This is complementary to our algorithm, which scales efficiently (i.e., polynomially) in particle number and energy. In particular, we suspect that some of their techniques may be applicable to our algorithm for BosonSampling states to reduce the dependency on the energy. Finally, our learning algorithm has an exponential runtime speedup compared to [14], which learns so-called "Gaussian-entanglable" states in modest sample complexity but exponential time complexity using classical shadows.

Techniques and proof sketch

Symplectic invariants and Gaussian convertibility. Given a bosonic quantum state, we construct many quantities that are invariant under the application of a general Gaussian unitary. These invariants are constructed from functions of the moments of the state that are invariant under symplectic transformations. This yields a necessary condition for two states to be *Gaussian convertible*—if two states are related by a Gaussian unitary, then all of their symplectic invariants must agree.

In more detail, letting $\boldsymbol{r} = (x_1, \ldots, x_n, p_1, \ldots, p_n)$ denote the canonical position and momentum operators on an n mode system, we define the *moment tensors* of a state ρ to be

$$\Sigma^{(t)}_{i_1,\ldots,i_t} = \mathrm{Tr}[\rho \tilde{r}_{i_1} \ldots \tilde{r}_{i_t}], \qquad \tilde{r}_i = r_i - \mathrm{Tr}[\rho r_i]. \tag{2}$$

In Gaussian bosonic quantum information, $\Sigma^{(2)}$ is the covariance matrix; the covariance matrix, along with the mean displacements $\mathrm{Tr}[\rho r_i]$, fully specify a Gaussian state ρ. A general Gaussian unitary acts as a displacement and as a symplectic transformation $S \in \mathrm{Sp}(2n)$. Gaussian states can be classified up to symplectic equivalence (*i.e.* up to a Gaussian unitary) by their symplectic eigenvalues via Williamson's decomposition [19].

In this work, we take the first step towards generalizing this classification beyond Gaussian states. Namely, using classical invariant theory [20–23], we enumerate a generating set of all symplectic invariants that are polynomials in the moment tensor entries. This yields an promising way to find witnesses that disprove Gaussian convertibility. Specifically, if two arbitrary states ρ and σ are related by a Gaussian unitary, then all of their symplectic invariants are the same. Finding a single invariant that is different between the two then suffices to prove that they are not Gaussian convertible.

G_t states. A state is Gaussian if it is the thermal or ground state of a Hamiltonian that is quadratic in the quadrature operators. From this, it follows that Gaussian states are fully specified by their first and second moments. In this work, we introduce a new class of states, called G_t states, that generalize beyond Gaussian states. Specifically, for even t, a G_t state is a thermal state or ground state of a non-degenerate Hamiltonian that is at most degree t in the quadrature operators. For example, the Fock state $|\boldsymbol{f}\rangle$ is non-Gaussian and is therefore not a G_2 state. However, because it is the ground state of

the quartic Hamiltonian $H = \sum_{i=1}^{n} (\hat{n}_i - f_i)^2$ build from the number operators $\hat{n}_i$ which are themselves quadratic, it is a G_4 state.

Next, we show that a G_t state is fully specified by its first t moment tensors—that is, $\mathrm{Tr}[\rho r_i]$ along with $\Sigma^{(s)}$ for $s \leq t$. This means that, given copies of a state that is promised to be G_t, if one could fully measuring all moments up to t^{th} order, then one has in principle learned all the information about the state. Even so, it is highly nontrivial to determine to what precision one needs to measure the moments in order to accurately learn the state (see e.g. [16]).

Algorithm, proof of correctness, and error analysis. Next, we show the utility of considering G_t states in learning theory. Because the set of G_t states is closed under Gaussian unitaries, and because Fock states are G_4 states, it follows that any BosonSampling state $|\psi\rangle = \mathcal{U}\,|\boldsymbol{f}\rangle$ is a G_4 state and is therefore fully specified by its second and fourth moments (note its first and third moments vanish). Given polynomially many copies of $|\psi\rangle$, the second and fourth moments can be learned to inverse polynomial precision by making Gaussian measurements on the copies (see manuscript for details). Thus, we can learn the moment tensors $\Sigma^{(2)}$ and $\Sigma^{(4)}$ to precision $1/\operatorname{poly}(n)$ with $\operatorname{poly}(n)$ measurements. We then construct an explicit polynomial time algorithm that learns $|\psi\rangle$ to $1/\operatorname{poly}(n)$ fidelity from the moment tensors, therefore arriving at Theorem 2. We again emphasize that explicit error bounds and runtimes are provided in the manuscript.

We now sketch the proof of Theorem 2 for learning the state $|\psi\rangle = \mathcal{U}\,|\boldsymbol{f}\rangle$. We first consider the error free case where the second and fourth moments are known exactly. We construct an algorithm that uses various linear algebra routines on the moment tensors to perfectly learn the state. For simplicity, assume that $\boldsymbol{f}$ is ordered in increasing order, so that $b_1 = f_1 = \cdots = f_{\ell_1}$, $b_2 = f_{\ell_1+1} = \cdots = f_{\ell_1+\ell_2}$, etc. First, we learn a unitary $\mathcal{W}$ that splits the state into blocks so that $\mathcal{W}\,|\psi\rangle = (\mathcal{W}_1\,|b_1, \ldots, b_1\rangle) \otimes (\mathcal{W}_2\,|b_2, \ldots, b_2\rangle) \otimes (\ldots)$. We then learn each $\mathcal{W}_i$ via an algorithm that we describe in the manuscript. Intuitively, the second moments of the state are used to find $\mathcal{W}$ in order to block diagonalize, and the fourth moments are then used to learn each $\mathcal{W}_i$. Crucially, the high degeneracy in the second moments makes fourth moments necessary. To perform the error analysis, we use various techniques in matrix analysis [24] and perturbation bounds for symplectic diagonalization [25, 26]. A notable feature of our proof is how we deal with the aforementioned degeneracy. $\mathcal{W}$ is highly non-unique, and when there is an error in our knowledge of the moments, the $\mathcal{W}$ that the algorithm finds will not perfectly split the state into blocks. We show that the algorithm does not care *which* block diagonal state we end up close to, but instead how close we are to *any* block diagonal state. Thus, we prove an upper bound on the distance to the nearest block diagonal state.

We emphasize that given the moment tensors, the full algorithm in Theorem 2 runs efficiently and only uses basic linear algebra routines—namely, matrix diagonalization, singular value decomposition, and Williamson decomposition.

Broad impact

Beyond the specific results we prove, our work can be thought of as demonstrating the utility of moment theory for bosonic states. Our concept of G_t states is a natural generalization of the ubiquitous Gaussian states, and we show the applicability of this concept via explicit learning algorithms for G_4 states. Indeed, we strongly suspect that variants of our algorithm utilizing higher moments will work for more general G_t states. Furthermore, the invariants offer a concrete step in the direction of a full generalization of Williamson's theorem, which would specify a *canonical form* of G_t states and could find much use in resource theories where Gaussian operations are considered "free" [27]. Finally, the definition of G_t states can be extended to fermions, thus encompassing interesting states found in quantum chemistry and many-body physics. The moment methods that we develop in this work can likely be applied to such fermionic states as well.

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