Fermionic Insights into Measurement-Based Quantum Computation: Circle Graph States Are Not Universal Resources

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Background. Measurement-based quantum computation (MBQC) is a promising route to develop practical, large-scale quantum computers [BBDRV09]. In MBQC, the computation proceeds by first preparing an entangled resource state, and then performing a sequence of adaptive single-qubit measurements. Whether or not universal quantum computation can be achieved in this manner depends on the choice of resource state.

Formally, a family of states $\mathcal{F}$ is said to be universal for MBQC if, for every k-qubit state $|\varphi\rangle$, there exists an n-qubit state $|\psi\rangle \in \mathcal{F}$ $(n \geq k)$ such that $|\varphi\rangle|+\rangle^{\otimes(n-k)}$ can be obtained from $|\psi\rangle$ using only local operations and classical communication (LOCC). Considerable effort is therefore made towards identifying when a family of resource states is universal. For example, Van Den Nest *et al.* [VMDB06] showed that a necessary condition for universality is that the entanglement-width of the family $\mathcal{F}$ is unbounded.

Main results. In this work, we restrict our attention to graph states, wherein entanglement-width is equivalent to the rank-width of the underlying graph. Geelen *et al.* [GKMW23] showed that the rank-width of a family of graphs is unbounded if and only if it includes all circle graphs as vertex minors. Circle graphs are defined as the intersection graph of a chord diagram: vertices correspond to chords, and an edge is drawn whenever two chords cross. If it could be shown that the family of circle graph states is universal for MBQC, then this result would provide a complete characterization of MBQC universality under stabilizer resource states. However, we in fact resolve this open question in the negative.

Theorem (Informal)**.** *Circle graph states are not universal for MBQC, unless* $\mathsf{BQP} = \mathsf{BPP}$.

This is significant for two main reasons. First, circle graphs are closed under vertex minors, which by the rank-width property of Geelen *et al.* [GKMW23] implies that the family of circle graph states has unbounded entanglement-width. Thus we give another concrete example of a family of graph states with nontrivial entanglement which is nonetheless not universal. The choice of circle graphs is also a natural graph-theoretic construction. In contrast, some prior examples of efficiently simulable resource states have limited entanglement-width by definition [VDVB07]. The closest related example to our result is due to Bravyi and Raussendorf [BR07], upgraded in a follow-up work [BGL22], showing that surface code states are not efficiently universal in a similar vein.

Second, Dahlberg and Wehner [DW18] showed that two graph states can be transformed into each other by local Clifford operations and classical communication (LCOCC) if and only if their corresponding graphs are vertex minors of each other. In other words, taking vertex minors is equivalent to LCOCC. This eventually implies that *any* family with unbounded entanglement-width must be able to reach *all* circle graph states via LCOCC. Informally, this means that every universal family of stabilizer states essentially contains all circle graph states, up to quantumly trivial transformations. Yet, our no-go result implies that the circle graphs by themselves are not sufficient for universality. Identifying that "missing ingredient" is the holy grail of MBQC universality, and our work rules out one seemingly promising direction.

Techniques. We feel our techniques are at least as important as the MBQC result itself. We show that there exists a classical, polynomial-time algorithm for simulating MBQC under circle graph

states. This is accomplished by identifying a surprising connection between circle graph states and fermionic Gaussian states (FGSs) on a larger space. In turn, FGSs can be manipulated efficiently by techniques related to free-fermion solvability, which enables our claimed classical simulation algorithm.

To map between fermions and qubits, we employ Kitaev's mapping which was introduced to solve the honeycomb model [Kit06]. This mapping is succinctly described in Table 1. In brief, for each qubit $j \in [n]$ we introduce four Majorana operators $c_j^1, c_j^2, c_j^3, c_j^4$. Products of pairs of these operators almost satisfy the single-qubit Pauli algebra, but they live in a 4-dimensional Hilbert space; we identify a qubit by projecting into the $+1$-eigenspace of the operator $D_j = -c_j^1 c_j^2 c_j^3 c_j^4$, where one may show we indeed recover the correct Pauli algebra.

	c^1	c^2	c^3	c^4
c^1	$\mathbb{I}$	iZ	$-iY$	iX
c^2	$-iZ$	$\mathbb{I}$	iX	iY
c^3	iY	$-iX$	$\mathbb{I}$	iZ
c^4	$-iX$	$-iY$	$-iZ$	$\mathbb{I}$

Table 1: The "multiplication table" for mapping four Majorana operators to a single qubit.

Next we define a class of free-fermion (Gaussian) states on these Majoranas. Consider a graph where we associate each qubit j to a supernode of 4 half-edges h_j^α, $\alpha \in [4]$. Each half-edge is then connected to one of the 4 half-edges on some other qubit k. This yields a 4-regular multigraph on supernodes, or equivalently a perfect matching on $4n$ half-edges. Then if we associate each half-edge to a Majorana operator, this perfect matching naturally induces an FGS that we call a *matching state*:

$$|\Psi\rangle\langle\Psi| = \prod_{(h_j^\alpha, h_k^\beta) \in \mathrm{PerfMatch}(4n)} \left(\frac{\mathbb{I} - i c_j^\alpha c_k^\beta}{2} \right). \tag{1}$$

We show that such a state is a uniform superposition over a set of 2^{n-1} stabilizer states, each of which is encoded into the joint s_j-eigenspace ($s_j \in \{\pm 1\}$) of the operators $D_1, \ldots, D_n$. We can therefore single out one particular stabilizer state by projecting $|\Psi\rangle$ onto the joint $+1$-eigenspace, which we recall is our identified n-qubit space.

It turns out that this projected-out stabilizer state is precisely the graph state $|G\rangle$ of some circle graph G. This largely follows from a classic result in graph theory: any circle graph can be constructed from the Eulerian tour of some 4-regular multigraph [Bou89]. An Eulerian tour is a closed walk through a graph such that every edge is traversed exactly once. This fact enables us to take any matching state and identify an encoded n-qubit circle graph state. We can also efficiently move in the reverse direction, borrowing off of algorithms for circle graph recognition [GPTC14].

Although the graph theory correspondence already exists, there still remains some work to show that the states actually behave how we expect them. For example, we need to show how to construct circle graph stabilizers from the perfect matching, and show that the appropriate sequence of Majorana operators indeed correctly project down to the n-qubit stabilizers under the fermion-to-qubit mapping. Recall that the stabilizer generators of a graph state $|G\rangle$ are $S_j = X_j \prod_{k \in \mathcal{N}_G(j)} Z_k$. The high-level idea is as follows. Draw an Eulerian tour on the 4-regular multigraph. Then at each node j, we split the tour into two cycles C_j^1, C_j^2. To each cycle C_j^a we define the operators $\tilde{S}_j^a$ as the product of Majorana operators (half-edges) traversed along the cycle. One can show that, under

the mapping of Table 1, both $\tilde{S}_j^1, \tilde{S}_j^2 \mapsto S_j$ within the n-qubit subspace. Specifically, we can track the behavior of each cycle and which Majoranas it traverses, and explicitly show that on each node k we either get the operator X, Z, or $\mathbb{I}$.

It remains to show how these pieces all fit together to give an efficient simulation of MBQC. To simulate MBQC, we appeal to the gate-by-gate algorithm of Bravyi, Gosset, and Liu [BGL22]. They show that MBQC with a resource state $|\psi\rangle$ can be classically simulated provided two efficient subroutines: (1) sampling $x \in \{0,1\}^n$ from the distribution $|\langle x|\psi\rangle|^2$, and (2) computing $|\langle x|(U_1 \otimes \cdots \otimes U_n)|\psi\rangle|^2$ for any sequence of single-qubit unitaries U_j. In our setting with $|\psi\rangle = |G\rangle$ a circle graph state, (1) is standard by stabilizer-simulation methods [AG04]. We show how to perform (2) below.

First, we show that any single-qubit state $|\phi_j\rangle$ can represented in the fermionic space by a state $|\tilde{\phi}_j\rangle$ on four Majorana modes, whose expectations agree with the Bloch vector of $|\phi_j\rangle$. For example, $\langle\tilde{\phi}_j|(-ic_j^1c_j^4)|\tilde{\phi}_j\rangle = \langle\tilde{\phi}_j|(-ic_j^2c_j^3)|\tilde{\phi}_j\rangle = \langle\phi_j|X|\phi_j\rangle$. Furthermore, we prove that this state has all the properties we desire: it is Gaussian, it is a $+1$-eigenstate of D_j, and its projection onto that eigenspace is precisely $|\phi_j\rangle$. Extending to product states over n qubits is straightforward: define the state

$$|\tilde{\Phi}\rangle\langle\tilde{\Phi}| = \prod_{j\in[n]} |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j| \tag{2}$$

which is also Gaussian and lives in the joint $+1$-eigenspace of $D_1, \ldots, D_n$.

Because the projection of the matching state $|\Psi\rangle$ onto the same subspace is $\frac{1}{\sqrt{2^{n-1}}}|G\rangle$, we immediately see that

$$|\langle\Phi|G\rangle|^2 = 2^{n-1}|\langle\tilde{\Phi}|\Psi\rangle|^2. \tag{3}$$

But $|\tilde{\Phi}\rangle$ and $|\Psi\rangle$ are both Gaussian, so a standard formula [BG17, Equation 22] allows us to express this overlap in terms of a Pfaffian of their $4n \times 4n$ fermionic Gaussian covariance matrices Γ_Φ, Γ_Ψ:

$$|\langle\tilde{\Phi}|\Psi\rangle|^2 = \frac{1}{4^n}\operatorname{pf}(\Gamma_\Phi + \Gamma_\Psi), \quad \text{where } [\Gamma_\Phi]_{ab} = -\frac{i}{2}\langle\Phi|[c_a, c_b]|\Phi\rangle, \text{ etc.} \tag{4}$$

The Pfaffian can be computed in $\mathcal{O}(n^3)$ time [Wim12; RW11] and hence every step in the framework of [BGL22] applied to circle graph states is efficient.

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